18th Annual Computational Neuroscience Meeting
CNS 2009, **poster number 252**

Frequency Shift in Phase Oscillators Driven by Colored Noise: Implications for Electronics and Neuroscience

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1. Introduction

It is well-known that the natural frequency of nonlinear oscillators can be affected when driven by a periodic force or when embedded in a network [1]. In contrast, the effects of stochastic forces on the frequency of nonlinear oscillators are incompletely understood. A thorough characterization of this phenomenon is nonetheless crucial for several fields, including electronics and neuroscience. Indeed, the implementation of radiofrequency devices requires precise electrical-oscillator circuits, in which thermal noise is an unavoidable perturbation that according to Nyquist's theorem [2] can be described as low-pass filtered white noise (colored noise). As for neuroscience, neuronal networks frequently display oscillations that enable the firing of an individual neuron only at a certain phase of the global rhythm [3]. This way, the network oscillation acts as a clock signal providing a temporal reference to encode, multiplex, and route information with millisecond precision in several areas of the brain [3]. For this neuronal code to be reliable, however, the robustness of global oscillations with respect to stochastic perturbations has to be demonstrated. Here, we take a step in this direction by describing how the mean frequency of a generic oscillator is affected by the correlation time and the amplitude of stochastic perturbations.

2. Results

We investigate the stochastic dynamics of a generic phase oscillator, φ with phase response, $Z(\varphi)$ driven by an Orstein-Uhlenbeck process, u (colored noise) with amplitude, σ whose autocorrelation function, C decays exponentially with time constant, τ:

$$\begin{cases} \dfrac{d\varphi}{dt} = \omega + \sigma Z(\varphi)u(t) \\ \dfrac{du}{dt} = -\dfrac{u}{\tau} + \sqrt{\dfrac{2}{\tau}}\eta(t) \end{cases} \qquad C(s) = \langle u(t)u(t-s)\rangle = \exp\left(-|s|/\tau\right)$$

We then calculate the mean frequency shift due to the effect of the colored noise as:

$$\langle \Delta\omega \rangle = \left\langle \frac{d\varphi}{dt} - \omega \right\rangle = \lim_{T\to\infty} \frac{\sigma}{T}\int_0^T Z(\varphi(t))u(t)dt$$

Assuming that the stochastic drive is weak ($\sigma<<\omega$), the mapping of the phase, φ onto time, t reads:

$$\varphi(t) \approx \omega t + \sigma\int_0^t Z(\omega s)u(s)ds$$

which allows us to calculate the mean frequency shift, thereby obtaining:

$$\langle \Delta\omega \rangle = \lim_{T\to\infty} \frac{\sigma^2}{T\omega}\int_0^T \frac{dZ(\omega t)}{dt}\int_0^t Z(\omega s)C(s-t)dsdt$$

Finally, by expanding the phase-response curve of the oscillator, $Z(\varphi)$ as a Fourier series:

$$Z(\varphi) = \sum_n A_n\cos(n\varphi) + B_n\sin(n\varphi)$$

and integrating, we arrive at:

$$\boxed{\langle \Delta\omega \rangle = -\omega\frac{\sigma^2}{2}\sum_n \frac{n^2\tau^2\left(A_n^2+B_n^2\right)}{1+n^2\omega^2\tau^2}}$$

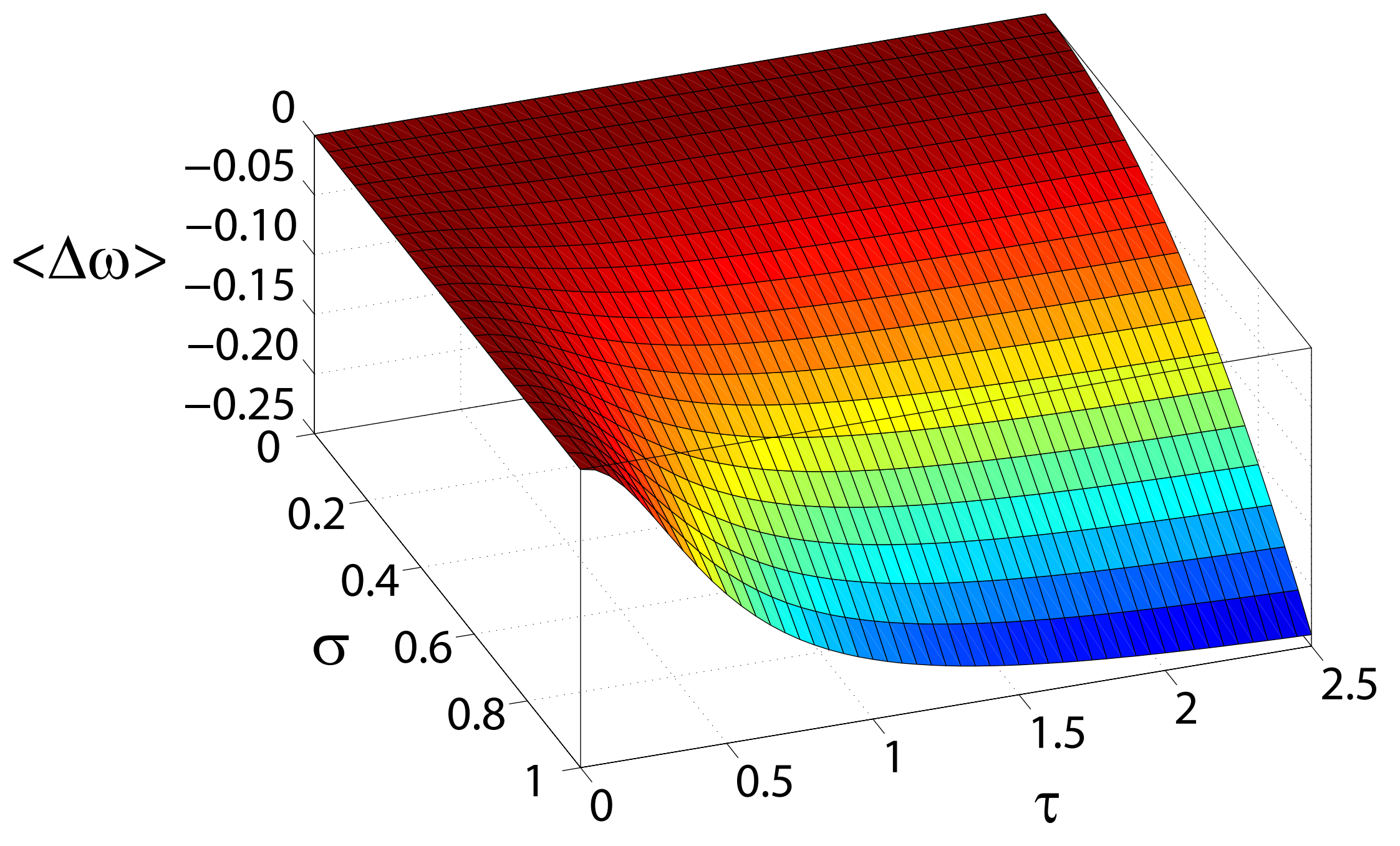

Figure 1: Mean frequency change as a function of both, the amplitude, σ and the autocorrelation time, τ of the colored-noise for ω=2 and a sinusoidal phase-response curve, $Z(\varphi) = -\sin(\varphi)$.

3. Example: Normal form of a supercritical Hopf bifurcation

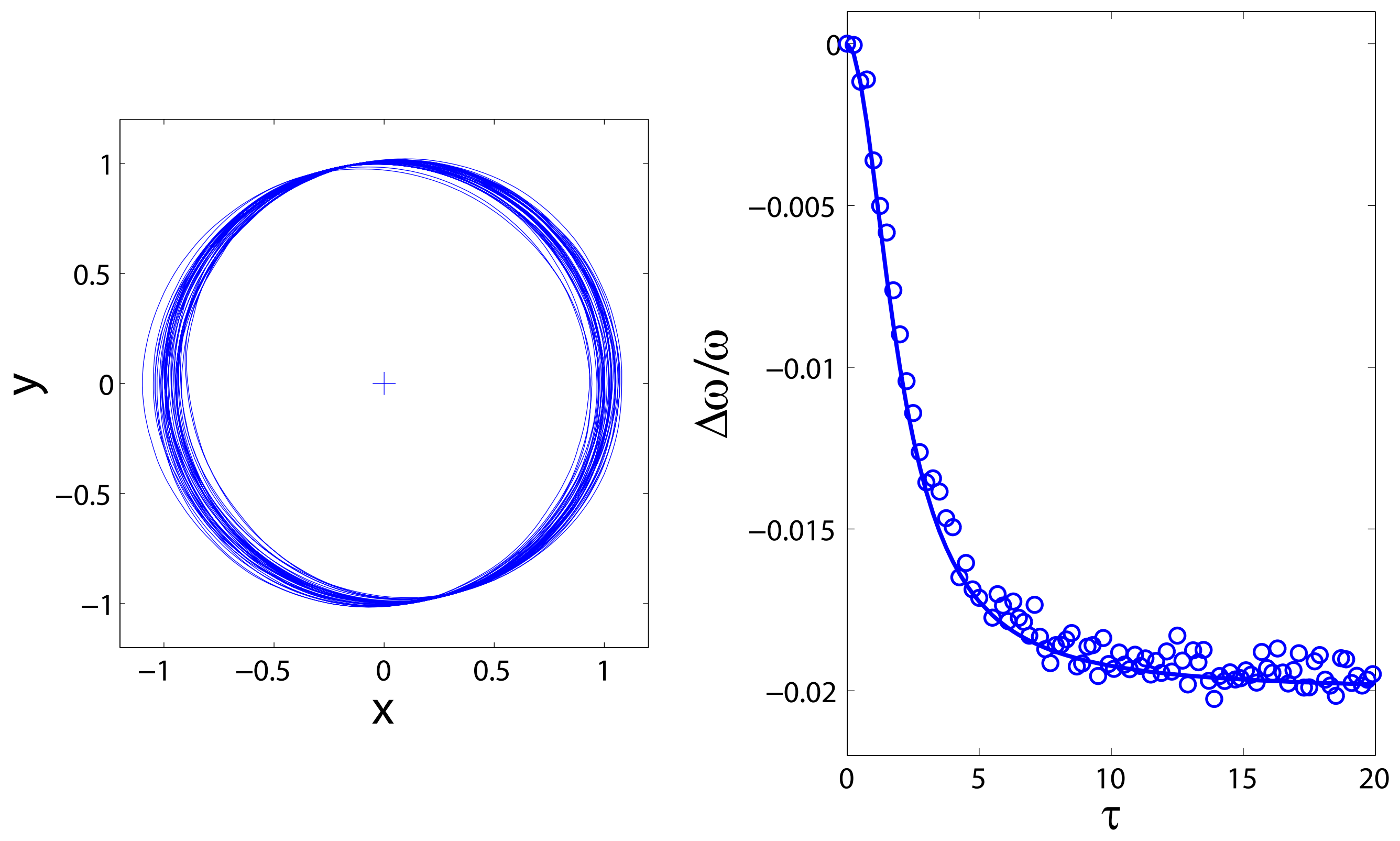

Figure 2: Frequency shift induced by colored noise in a generic model of sustained oscillations (normal form of a supercritical Hopf bifurcation). *Left*: Perturbed trajectories for colored noise with τ = 20. The cross indicates the origin (0,0) with respect to which the phase is calculated in polar coordinates. *Right*: Relative frequency shift as a function of the correlation time of the noise, τ. The data from simulations (circles) match the theoretical prediction (line).

4. Example: The van der Pol oscillator

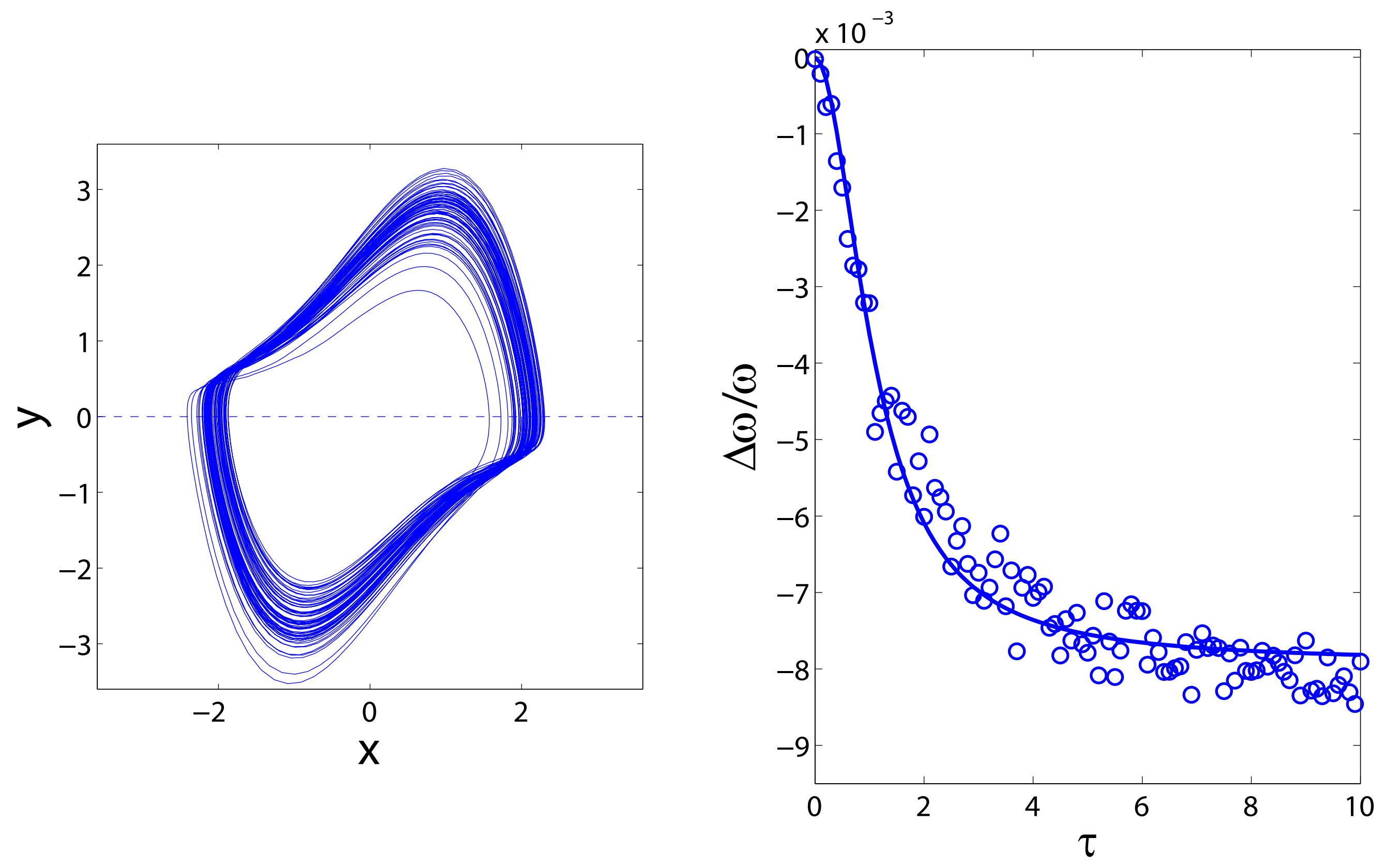

Figure 3: Frequency shift induced by colored noise in the FitzHugh-Nagumo oscillator. *Left*: Perturbed trajectories for colored noise with τ = 10. The dashed line indicates the Poincaré section with respect to which the instantaneous phase is calculated. *Right*: Relative frequency shift as a function of the correlation time of the noise, τ. The data from simulations (circles) fit the trend predicted by the theory (line).

5. Example: The FitzHugh-Nagumo model

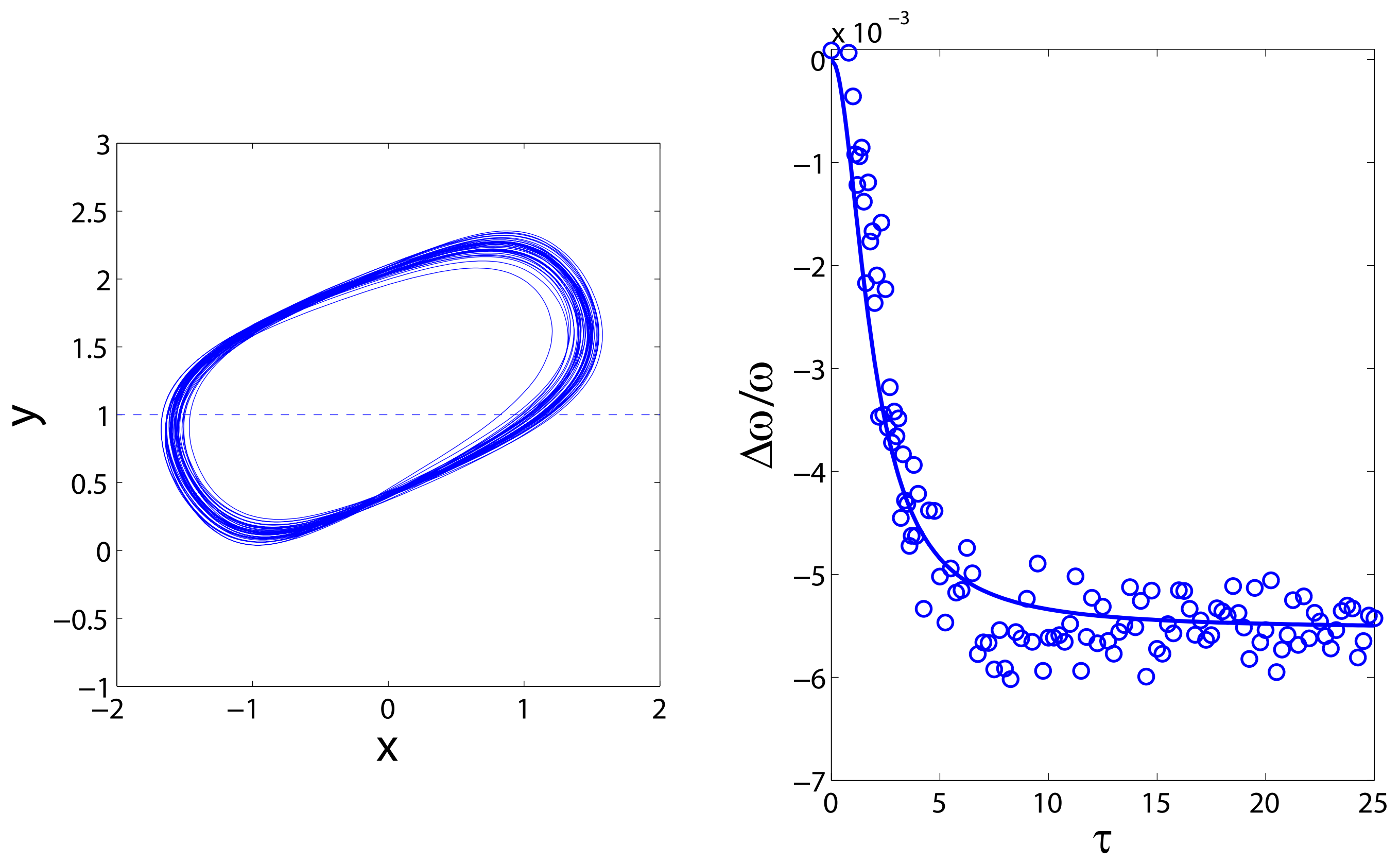

Figure 4: Frequency shift induced by colored noise in the FitzHugh-Nagumo oscillator. *Left:* Perturbed trajectories for colored noise with τ = 25. The dashed line indicates the Poincaré section with respect to which the instantaneous phase is calculated. *Right:* Relative frequency shift as a function of the correlation time of the noise, τ . The data from simulations (circles) fit the trend predicted by the theory (line).

Acknowledgments

Work supported by the Mount Sinai Health Care Foundation and the Alfred P. Sloan Foundation.

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