Math 319: Applied Probability and Stochastic Processes for Biology
Fall 2008. Fifth Assignment.

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Due Wednesday 12/3 in class.

1. Simulate a diffusion process without drift in one spatial dimension as follows. On each time step,

   - Generate a random integer n from the Poisson distribution with mean $\mu = 10$.
   - Place n molecules at location $x = 0$ in a domain $-50 \leq x \leq 50$.
   - Let each molecule in the domain increment its location via $x \to x + bz$ where z is normally distributed with mean zero and unit variance, and b is a parameter. Use $b = 1$ initially – you can change this to see how it affects the behavior.
   - If any molecules lie outside the range $-50 \leq x \leq 50$, remove them from the simulation.
   - Repeat until the system comes more or less to equilibrium. (Estimating how long this might take is part of the problem; you can use trial and error or consider the properties of the random walk or both.)

   Questions:

   (a) If $c(x, t)$ represents the local density of molecules at location x and time t, write down the differential equation that approximately describes how $c(x, t)$ should behave. Be sure to include appropriate boundary conditions and initial conditions. And identify any constants (e.g. how does the constant D in the diffusion equation relate to b)?

   (b) Based on the differential equation in (a), predict the steady-state distribution of your simulated molecules at long times.

   (c) Compare the actual distribution with your prediction by calculating the local discrete densities in bins of size 5, i.e. count the numbers of molecules in each region $I_{-45} = -50 \leq x < 40$, $I_{-35} = -40 \leq x < 30$, $\cdots$, $I_{45} = 40 \leq x < 50$. Plot the prediction and the results of several (≥ 10) simulations together.

   (d) Derive an expression for the variance of the numbers of molecules observed in each of your sampling regions defined above.

   (e) Compare the actual variance (across ≥ 10 simulations) with your prediction, and plot the comparison.

2. Plot a linearly interpolated sample path over the interval $0 \le t \le 1$ of the Ito sums approximation

$$S_n = \sum_{j=1}^{n} f(t_j) \left[W_{t_{j+1}} - W_{t_j}\right]$$

for the Ito stochastic integral

$$\int_0^t W(s)\, dW(s) = \frac{1}{2}\left[W(s)^2 - t\right]$$

calculated with equidistant steps $\Delta t = 0.01$. Compare the results with (a) the corresponding sample path of the exact solution and (b) the Stratonovich sums approximation

$$S_n^{\text{Strat}} = \sum_{j=1}^{n} f\left(\frac{t_{j+1} + t_j}{2}\right) \left[W_{t_{j+1}} - W_{t_j}\right].$$

3. Euler's method for approximate numerical integration of a differential equation is called the "Euler-Maruyama Method" when applied to a stochastic differential equation. Given an SDE

$$dx = a(x,t)dt + b(x,t)dW$$

we set a time step size h and let x evolve so that

$$x_{n+1} = x_n + a(x,t)h + b(x,t)\sqrt{h}\xi_n$$

where the ξ_n are normally distributed random variables with mean zero and variance one. Use this method to generate approximate numerical solutions of the leaky integrate-and-fire neuron model with additive noise

$$dv = (-\gamma v + I(t))\, dt + \sigma\, dW(t).$$

Use $v = 0$ as the reset potential after the model fires a "spike"; assume the spiking threshold is $v = 1$.

(a) Let $\gamma = 0$ and $I = 0$. Run 10,000 trials with a time step of h=0.01, for a long enough time so that a substantial number of the trials reach threshold at least twice. Use $\sigma = 0.1$. Compare the histogram of first passage times with that predicted analytically. Plot the first and second interspike intervals against each other and comment on whether the scatter plot appears as you expected.

(b) Let $\gamma = -1$ and $I = 1.1$ represent a constant current and membrane leakage. Run for long enough to collect five spikes on a large number of trials. Find the interspike interval (ISI) distributions and plot them. Also plot the histogram for the absolute time of the first spike, the 2nd spike etc., (superimposed is OK.) Plot the standard deviation of the spike time distribution versus spike number and explain its shape.

(c) Let $\gamma = -1$ and $I = 1.1 + .1\sin(\omega t)$. Find the ISI distribution as in (b), plot the histograms for the absolute time of the first, second spike etc., and plot the standard deviation. Explain the difference between this case and case (b). Create a return plot for successive ISIs (plot ISI n versus ISI $n+1$ as a scatter plot). Comment on what you observe. Find the frequency ω at which the effects of periodic stimulation are most pronounced.