Normal forms in Lorentzian Spaces

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0 Introduction

The well-know spectral theorem in linear algebra ([G], p. 222) may be formulated as saying that given a self-adjoint transformation S of $\mathbf{R}^n$, there is an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors of S. A somewhat less well-known theorem ([G], p. 230) gives normal forms for skew-adjoint transformations: there is an orthonormal basis with respect to which a skew-adjoint transformation A satisfies the conditions: $Ae_i = 0, i > 2p; Ae_{2i-1} = -\chi_i e_{2i}, Ae_{2i} = \chi_i e_{2i-1}, i \leq p$. A restatement of the above is that S can be represented by a diagonal matrix and that A can be represented by matrix which is block diagonal with p 2 $\times$ 2 blocks of the form:

$$\begin{bmatrix} 0 & \chi_i \\ -\chi_i & 0 \end{bmatrix}.$$

Note that the result for the skew adjoint transformation A follows easily from the first theorem by using the fact that $A^2 = S$ is self-adjoint.

In this note we solve the analogous problem when $\mathbf{R}^n$ is replaced by $\mathbf{R}^{n,1}, (n+1)$ - dimensional Lorentzian space. Our motivation for considering this problem comes from the theory of Lie groups and its application to Hamiltonian systems. If G is a Lie group, then it acts as linear transformations of its Lie algebra $\mathcal{G}$ by the adjoint action and on the dual space $\mathcal{G}^*$ by the co-adjoint action. When G is a matrix group, the Lie algebra is a matrix Lie algebra, and the action of G on $\mathcal{G}$ is just conjugation. In general, the co-adjoint action is different from the adjoint action [F, p. 41], but when the Lie group is semi-simple there is a natural isomorphism between $\mathcal{G}$ and $\mathcal{G}^*$, and the co-adjoint action may also be regarded as conjugation. The orbits of the co-adjoint action are symplectic manifolds (Kyrillov's Theorem – see [F, p. 43]) and form a natural setting for the study of Hamiltonian systems with symmetries.

Now in the case where $G = \mathrm{SO}(n, \mathbf{R})$, the group of orientation-preserving orthogonal transformations of $\mathbf{R}^n$, the Lie algebra $\mathcal{G} = \mathrm{so}(n)$ is just the algebra of skew-symmetric matrices. The normal forms theorem gives canonical representatives of the orbits of the (co-)adjoint action.

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Our goal is to give a similar characterization of the orbits of the adjoint action of the semi-simple Lie group $\mathrm{SO}^+(n,1)$ on its Lie algebra $\mathrm{so}(n,1)$.Note that the group $\mathrm{SO}^+(n,1)$ is isomorphic to the group of rigid motions of *hyperbolic* n - dimensional space, and its Lie algebra is the space of infinitesimal rigid motions (or *Killing fields*) on hyperbolic space.

1 Basic definitions

Lorentzian $n+1$ space is the $(n+1)$ - dimensional real vector space spanned by vectors $e_1, \ldots, e_n, e_{n+1}$, equipped with the bilinear form $\langle\ ,\ \rangle$ given by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j \leq n \\ -1 & i = j = n+1 \\ 0 & i \neq j \end{cases}$$

(The name *Minkowski space* or *Minkowski space-time* is usually reserved for the case $n = 3$.). A vector v in V is called **isotropic** or **null** if $\langle v, v \rangle = 0$.

If V is a real vector space with a bilinear form $\langle\ ,\ \rangle$, a linear transformation

- $S : V \to V$ is **self-adjoint** if $\langle Sv, w \rangle = \langle v, Sw \rangle$ for all vectors v, w in V;
- $A : V \to V$ is **skew-adjoint** if $\langle Av, w \rangle = -\langle v, Aw \rangle$ for all vectors v, w in V; and
- $g : V \to V$ is **orthogonal** if $\langle gv, gw \rangle = \langle v, w \rangle$ for all v, w in V.

In the case that V is Lorentzian $n+1$ space, the orthogonal transformations of V form the Lie group $\mathrm{O}(n,1)$; the subgroup $\mathrm{SO}^+(n,1)$ consists of those transformations which preserve the orientation of V and carry each component of $\{v \mid \langle v, v \rangle = -1\}$ to itself. The skew-adjoint transformations form its Lie algebra $\mathrm{so}(n,1)$ (For an exposition of the properties of orthogonal groups, see [P],[M], or [H]).

Let J be the diagonal matrix

$$\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}.$$

Then we may describe the matrix representations of the transformations defined in the previous paragraph in terms of the standard basis.

- The matrix of a self-adjoint transformation satisfies $S^\top = JSJ$;
- the matrix of a skew-adjoint transformation satisfies $A^\top = -JAJ$; and
- the matrix of an orthogonal transformation satisfies $g^\top Jg = J$, or $g^{-1} = Jg^\top J$.

A subspace W of V is called **non-singular** if V is the direct sum of W and its orthogonal complement $W^{\perp}$. This condition is equivalent to the statement that the bilinear form $\langle\,,\,\rangle$ when restricted to W remains non-singular. For any subspace W, $\dim(W) + \dim(W^{\perp}) = n+1$ [H, p.26] , so the condition of being non-singular is equivalent to the statement: $W \bigcap W^{\perp} = \{0\}$. A transformation with no non-singular invariant subspaces other than $\{0\}$ and the whole space is said to be an **indecomposable** transformation.

2 Self-adjoint transformations

If $S : V \to V$ is a self-adjoint linear transformation of Lorentzian $n+1$ space, and if $S(W) \subset W$ for some non-singular subspace W of V with $0 < dim(W) < n+1$, then $S(W^{\perp}) \subset W^{\perp}$, and we can decompose S into the direct sum of two transformations each of which is self-adjoint. One of the two spaces will be positive-definite, while the other will be a Lorentzian $k+1$ space for some $k < n$. Thus the normal forms theorems for Euclidean spaces, together with induction arguments allow us to restrict our attention to indecomposable transformations.

The following easily proved lemma is used in the proof of the theorem ([R]).

Lemma 2.1 *No two linearly independent isotropic vectors are orthogonal. Given an isotropic vector v, any vector w orthogonal to v is either a multiple of v or is such that $\langle w, w\rangle > 0$.*

Theorem 2.2 *Let $S : V \to V$ be a self-adjoint transformation of $V = \mathbf{R}^{n,1}$ which is indecomposable. Then $n \leq 2$, and if $n > 0$ there is an orthonormal basis for V with respect to which the matrix of S has one of the following forms:*

$$a)\quad \begin{bmatrix} \alpha & -\gamma \\ \gamma & \beta \end{bmatrix} \text{ with } |\, 2\gamma \,| > |\, \alpha - \beta \,|,$$

$$b)\quad \begin{bmatrix} \alpha + r & -\alpha \\ \alpha & -\alpha + r \end{bmatrix},$$

$$c)\quad \begin{bmatrix} \alpha + r & \alpha \\ -\alpha & -\alpha + r \end{bmatrix},$$

$$d)\quad \begin{bmatrix} r & \alpha & -\alpha \\ \alpha & \beta + r & -\beta \\ \alpha & \beta & -\beta + r \end{bmatrix}, \text{ where } \alpha \neq 0 \text{ in cases } b, c, \text{ and } d.$$

Proof. Suppose S is indecomposable and self-adjoint.

First suppose further that S has a complex eigenvalue $\lambda = \alpha + \beta i$, where $\beta \neq 0$. By complexifying, we may find a complex eigenvector $v + iw$. This means that $S(v + iw) = (\alpha v - \beta w) + i(\alpha w + \beta v)$; that is, $Sv = \alpha v - \beta w$ and $Sw = \beta v + \alpha w$. Since S is self-adjoint and $\beta \neq 0$,it easily follows that $\langle v, v\rangle = -\langle w, w\rangle$. Let P be the plane spanned by v and w. Then P is an invariant subspace of S. Since S is indecomposable and P is not trivial, if we can show that P is non-singular, it will follow that $P = V$.

If $\langle v, v\rangle \neq 0$ then the Grammian matrix

$$\begin{bmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle v, w\rangle & \langle w, w\rangle \end{bmatrix}$$

has negative determinant and therefore the bilinear form is non-singular. If $\langle v, v\rangle = 0$, then the only way the grammian could be singular would be if $\langle v, w\rangle$ also vanished. Thus v and w would be linearly independent isotropic vectors, but this violates the lemma.Thus P is non-degenerate. Since the characteristic polynomial of S has a complex root, the matrix of S must have the form a) of the theorem.

Next assume that all eigenvalues of S are real. If λ is an eigenvalue and v is an eigenvector with eigenvalue λ, then the line spanned by v is an invariant subspace of S. If the dimension of V is greater than 1, the assumption that S is indecomposable implies that v is an isotropic vector, that is $\langle v, v\rangle = 0$. If w is another isotropic vector and $Sw = \mu w$ for some $\mu \neq \lambda$, then self-adjointness implies $\langle v, w\rangle = 0$. Again, this is impossible in $\mathbf{R}^{n,1}$ as it violates the lemma. We conclude that S has only one eigenvalue r. Replacing S by the self-adjoint transformation $S - rI$, we may consider transformations whose only eigenvalue is 0.

ker(S) is exactly one-dimensional, for otherwise we could find a non-isotropic vector in the kernel. S is nilpotent, and we can choose $v_i, 0 \leq i \leq n$, such that $S(v_{i+1}) = v_i$ for $i \geq 1, S(v_0) = 0$,and v_0 is an isotropic vector.

Using self-adjointness, we have $\langle v_0, v_1\rangle = \langle v_0, Sv_2\rangle = \langle Sv_0, v_2\rangle = 0$. The lemma then implies that $\langle v_1, v_1\rangle > 0$.

The cases $n = 1$ and $n = 2$ remain since if $n \geq 3$ then we would get the following contradiction:

$$0 = \langle Sv_0, v_3\rangle = \langle v_0, Sv_3\rangle = \langle v_0, v_2\rangle = \langle Sv_1, v_2\rangle = \langle v_1, Sv_2\rangle = \langle v_1, v_1\rangle \neq 0.$$

If $n = 1$, then let $\{u_1, u_2\}$ be any orthonormal basis. Since S is nilpotent, im(S) = ker(S), so $S(u_1) = \alpha(u_1 \pm u_2)$ for some $\alpha \neq 0$. This leads to the cases b) and c) above.

As our final case, assume that $n = 2$. We may scale the v_i so that $\langle v_1, v_1\rangle = 1$. Choose an orthonormal basis $\{u_1, u_2, u_3\}$ for V such that v_1 is the first vector in the basis. Then $Su_1 = v_0 = \alpha(u_2 \pm u_3)$ for some $\alpha \neq 0$. From the equation, $0 = Sv_0 = \alpha(Su_2 \pm Su_3)$, we deduce that the matrix of S has one of the forms:

$$\begin{bmatrix} 0 & \alpha & -\alpha \\ \alpha & \beta & -\beta \\ \alpha & \beta & -\beta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \alpha & \alpha \\ \alpha & \beta & \beta \\ -\alpha & -\beta & -\beta \end{bmatrix}.$$

These are conjugate by $-J$. This is case d) with $r = 0$. □

As a result of theorem, we have the following results for normal forms for self-adjoint transformations of $\mathbf{R}^{n,1}$. The case $n = 1$ is exceptional and could also easily be proved directly without the machinery of the theorem.

Corollary 2.3 *Every self-adjoint transformation of $\mathbf{R}^{1,1}$ is conjugate by an element of $SO^{+}(1,1)$ to one whose matrix is either diagonal or a matrix of one of the following five types:*

$$\text{i.)}\quad \begin{bmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{bmatrix} \text{ with } \alpha \neq 0,$$

$$\text{ii.)}\quad \begin{bmatrix} 1+r & -1 \\ 1 & -1+r \end{bmatrix}, \qquad \text{iii.)}\quad \begin{bmatrix} -1+r & 1 \\ -1 & 1+r \end{bmatrix},$$

$$\text{iv.)}\quad \begin{bmatrix} 1+r & 1 \\ -1 & -1+r \end{bmatrix}, \text{ or } \quad \text{v.)}\quad \begin{bmatrix} -1+r & -1 \\ 1 & 1+r \end{bmatrix}.$$

No two of the six types of matrices are conjugate by an element of $SO^{+}(1,1)$.

Proof: From the theorem it follows that the matrix of any self-adjoint transformation of $\mathbf{R}^{1,1}$ is conjugate by an element of $\mathrm{SO}^{+}(1,1)$ to one whose matrix is either diagonal or one of types a), b), or c) as listed in the theorem. We note that the orthonormal bases chosen in the proof can always be chosen to preserve the orientations of V. We will first show that no two of these four types are conjugate. Since the dimensions of all invariant subspaces are preserved by conjugacy, we need only check that matrices of types b) and c) are not conjugate. This computation is straightforward. Take an arbitrary element

$$g = \begin{bmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{bmatrix} \text{ of } \mathrm{SO}^{+}(1,1). \text{ If } K = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

then an arbitrary matrix B of type b) will have the form $B = \alpha K + rI$, and an arbitrary matrix C of type c) will have the form $C = \alpha K^{\top} + rI$, where $\alpha \neq 0$. Hence we discover that $gBg^{-1} = \alpha gKg^{-1} + rI = \alpha(\cosh u + \sinh u)^2 K + rI$. Since $(\cosh u + \sinh u)^2 > 0$, gBg^{-1} is another matrix of type b). Similarly we find that gCg^{-1} is another matrix of type c). So types b) and c) are not conjugate.

Further, since $(\cosh u + \sinh u)^2 = e^{2u}$ is an arbitrary positive real number, we can find u so that $(\cosh u + \sinh u)^2 = \mid \alpha \mid^{-1}$. In case b), if $\alpha > 0$ and u is chosen appropriately, we obtain case *ii.*. If instead $\alpha < 0$, a similar choice of u leads to case *iii.)*. These are not conjugate to each other. Similarly for case c) we get cases *iv.)* when $\alpha > 0$ and *v.)* when $\alpha < 0$. These are also not conjugate to each other.

Finally, the condition $\mid 2\gamma \mid > \mid \alpha - \beta \mid$, implies that we could choose u such that $\tanh 2u = (\beta - \alpha)2\gamma$. If we conjugate a matrix A of type a) by the matrix g above, with this u, we obtain a matrix of the form:

$$\begin{bmatrix} \frac{\alpha+\beta}{2} & -\gamma' \\ \gamma' & \frac{\alpha+\beta}{2} \end{bmatrix}. \text{ This is a matrix of type } i.). \qquad \square$$

Corollary 2.4 *For $n > 1$, every self-adjoint transformation of $\mathbf{R}^{n,1}$ is conjugate by an element of $SO^+(n,1)$ to one whose matrix is either diagonal or a direct sum of a diagonal matrix and a matrix of one of the following five types:*

$$\text{i.)}\quad \begin{bmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{bmatrix} \text{ with } \alpha \neq 0,$$

$$\text{ii.)}\quad \begin{bmatrix} 1+r & -1 \\ 1 & -1+r \end{bmatrix}, \qquad \text{iii.)}\quad \begin{bmatrix} -1+r & 1 \\ -1 & 1+r \end{bmatrix},$$

$$\text{iv.)}\quad \begin{bmatrix} 0 & 1 & -1 \\ 1 & \gamma & -\gamma \\ 1 & \gamma & -\gamma \end{bmatrix} \text{ or,} \qquad v.)\quad \begin{bmatrix} 0 & -1 & 1 \\ -1 & \gamma & -\gamma \\ -1 & \gamma & -\gamma \end{bmatrix}.$$

No two of the six types of matrices are conjugate by an element of $SO^+(n,1)$.

Proof: From the theorem we conclude immediately that the matrix is either diagonal or a direct sum of a diagonal matrix and a matrix of type *a*), *b*), *c*), or *d*) as listed in the theorem, with the orthonormal bases chosen to preserve the orientations of V. As in the proof of Corollary 2.2, the dimensions of all invariant subspaces are preserved by conjugacy. Also as in Corollary 2.2, a matrix that is the direct sum of a diagonal matrix and a matrix of type *a*) is conjugate to a matrix that is the direct sum of the same diagonal matrix and a matrix of type *i.)*. So what remains to be shown is that matrices that are the direct sum of a diagonal matrix and matrices of types *b*) and *c*) are conjugate and that the direct sum of a diagonal matrix and a matrix of type *d*) is conjugate to the direct sum of a diagonal matrix and a matrix of type *iv*) listed above.

For the latter, conjugate a matrix of type *d*) by the matrix

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh u & \sinh u \\ 0 & \sinh u & \cosh u \end{bmatrix} \text{ and obtain ,}$$

$$(\cosh u + \sinh u) \begin{bmatrix} 0 & \alpha & -\alpha \\ \alpha & (\cosh u + \sinh u)\beta & -(\cosh u + \sinh u)\beta \\ \alpha & (\cosh u + \sinh u)\beta & -(\cosh u + \sinh u)\beta \end{bmatrix}.$$

We can find u such that $\cosh u + \sinh u = \mid \alpha \mid^{-1}$, and hence obtain a matrix of type *iv*) if we start with $\alpha > 0$ and of type *v*) if we start with $\alpha < 0$.

For matrices of types *b*) and *c*), define a diagonal matrix

$$K = \begin{bmatrix} (-1)^{n+1} & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix}.$$

Then K is an element of $SO^+(n,1)$. If B is the direct sum of a diagonal matrix and a matrix of type *b*) and C is the corresponding matrix that is the direct sum of a diagonal matrix and a matrix of type *c*), then $KBK^{-1} = C$. As in the preceding corollary, a matrix of type *b*) from the theorem will be conjugate to one of type *ii*), if $\alpha > 0$, and to one of type *iii*) if $\alpha < 0$. □

3 Skew-adjoint transformations

As in the case of self-adjoint transformations, if $A : V \to V$ is skew-adjoint, and if $A(W) \subset W$ for some non-singular subspace W of V with $0 < dim(W) < n+1$, then $A(W^\perp) \subset W^\perp$, and we can decompose S into the direct sum of two transformations each of which is self-adjoint. As before one of the two spaces will be positive-definite, while the other will be a Lorentzian $k+1$ space for some $k < n$. We will also need the following fact:

Lemma 3.1 *The characteristic polynomial of a skew-adjoint transformation is odd if dim(V) is odd and even if dim(V) is even.*

Proof. $\det(A - \lambda I) = \det(A^\top - \lambda I) = \det(-JAJ - \lambda I) = \det(-A - \lambda I) = (-1)^{n+1}$ $\det(A + \lambda I)$. □

Theorem 3.2 *Let $A : V \to V$ be a skew-adjoint transformation of $V = \mathbf{R}^{n,1}$ which is indecomposable. Then $n \leq 2$, and if $n > 0$ there is an orthonormal basis for V with respect to which the matrix of A has one of the following forms:*

$$a) \quad \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \text{with } \alpha \neq 0;$$

$$b) \quad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$c) \quad \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

*Proof:*If $\dim(V) = 2$, then the formula $A^\top = -JAJ$ implies immediately that the matrix of A has form *a*). We will next show that $\dim(V) \leq 3$ and classify the remaining case: $n = 2$.

Let $S = A^2$. Then S is self-adjoint. If $\dim(V) > 3$, then S has an eigenvector v with eigenvalue λ such that $\langle v, v \rangle = 1$. Let $w = Av$. Then $Aw = \lambda v$ and the subspace P spanned by v and w is invariant. If P is 1-dimensional, then v is an eigenvector of $A.P$ is non-singular. This contradicts indecomposability.

The above implies that $\dim(P) = 2$. If we were to assume that $\lambda \neq 0$, then $\langle w, w\rangle = \langle Av, w\rangle = -\langle v, Aw\rangle = -\lambda\langle v, v\rangle = -\lambda$. $\langle v, w\rangle = \langle v, Av\rangle = 0$, so P is non-singular. Since this contradicts indecomposability, we must have $\lambda = 0$ and w is isotropic.

Let $L = \ker(A)$. If $\dim(L) > 1$, then there is a vector x with $Ax = 0$ and w and x independent. Now $\langle x, w\rangle = \langle x, Av\rangle = -\langle Ax, v\rangle = 0$. Therefore, $\langle x, x\rangle \neq 0$ and the line spanned by x is nonsingular and invariant, contradicting indecomposability. Thus L is the line spanned by w. Since $\mathrm{im}(A) \subset L^{\perp}$ and both have codimension 1 in V, they are equal. In particular, $v = Au$ for some vector u. Now let W be the subspace spanned by u, v, and w. W is invariant, and we will see that it is nonsingular. This will show that if A is indecomposable then $\dim(V) \leq 3$.

$$\begin{aligned}
\langle u, v\rangle &= \langle u, Au\rangle = 0 \\
\langle u, w\rangle &= \langle u, Av\rangle = -\langle Au, v\rangle = -\langle v, v\rangle = -1 \\
\langle v, w\rangle &= \langle v, Av\rangle = 0 \\
\langle v, v\rangle &= 1 \\
\langle w, w\rangle &= 0
\end{aligned}$$

Putting these together, the matrix of the bilinear form restriced to W with respect to the basis u, v, w is the Grammian:

$$\begin{bmatrix} \langle u,u\rangle & \langle u,v\rangle & \langle u,w\rangle \\ \langle v,u\rangle & \langle v,v\rangle & \langle v,w\rangle \\ \langle w,u\rangle & \langle w,v\rangle & \langle w,w\rangle \end{bmatrix} = \begin{bmatrix} \langle u,u\rangle & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

which is non-singular. So if A is indecomposable, $n \leq 2$. If $n = 1$, then we have seen that A is a matrix of type a).

Suppose $\dim(V) = 3$. By Lemma 3.1, 0 is an eigenvalue of A. From the indecomposability of A we see that the corresponding eigenvector w must be isotropic and that the kernel is the line L spanned by w. Therefore, by the above arguement, $\mathrm{im}(A) = L^{\perp}$. But w is in $L^{\perp}$, so $w = Av$ for some v. $\langle w, v\rangle = 0$, so likewise $v = Au$ for some u. By Lemma 2.1, $\langle v, v\rangle > 0$, so we may scale these vectors so that $\langle v, v\rangle = 1$.

Let u_i be an orthonormal basis for V such that $u_1 = v$.A is skew- adjoint and so is of the form:

$$A = \begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{bmatrix}.$$

Since $Av = Au_1 = w$ is a non-zero isotropic vector, $\alpha = \pm\beta$. Since $Aw = 0$, $\gamma = 0$. As in the proofs of the Corollary 2.4 above, conjugation by the matrix h allows us to scale by $\mid \alpha \mid^{-1}$. This gives us the four cases: $A = \pm B_{\pm}$, where

$$B_{\pm} = \begin{bmatrix} 0 & -1 & \pm 1 \\ 1 & 0 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix}.$$

B_+ and B_- are conjugate via $-J$. B_+ and B_- are not conjugate. A must, therefore, be $\pm B_+$ which is type b) or c) of the statement of the theorem. $\square$

Corollary 3.3 *The matrix of any self-adjoint transformation of* $\mathbf{R}^{n,1}$ *can be put in the following block diagonal canonical form:*

$$\begin{bmatrix} 0 & & \\ & B & \\ & & C \end{bmatrix}$$

where 0 *is the* $m \times m$ 0 *matrix,* $m \geq 0$*,* B *is a block diagonal matrix with* 2×2 *blocks*

$$\begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}, 1 \leq i \leq p, p \geq 0,$$

and C *is one of the matrices described in Theorem 3.1. The representation is unique except for the order of the blocks in* B.

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