Math 402 additional problem for March 6, 2009

Recall from class that the associativity of tensor products means that there is a category with rings (with 1) for objects and (isomorphism classes of) bimodules for morphisms, with the tensor product for composition of morphisms. The following problem develops part of how this relates to the more familiar category of rings and homomorphisms.

In the following, all rings are assumed to have identity, all modules are assumed to be unitary, and all ring homomorphisms are assumed to map the identity to the identity.

Given rings R and S and a homomorphism $f : R \to S$, define an (R, S)-bimodule M_f as follows: M_f is the additive abelian group S, and for $m \in M_f = S$, $r \in R$, and $s \in S$,

$$rm = f(r) \cdot m \quad \text{and} \quad ms = m \cdot s,$$

where $\cdot$ denotes the multiplication operation of S.

1. Show that the above definition does in fact define an (R, S)-bimodule structure on M_f, and that if $R = S$ and $f = 1_R$ then this coincides with the natural (R, R)-bimodule structure on R.

2. Given ring homomorphisms $f : R \to S$, $g : S \to T$, prove that

$$M_f \otimes_S M_g \cong M_{gf}$$

as (R, T)-bimodules.

This implies that there is a functor from the category of rings and homomorphisms to the category of rings and bimodules that takes each ring R to itself and takes a homomorphism $f : R \to S$ to the (R, S)-bimodule M_f.