Math 402 homework for April 24, 2009

These are problems from chapter 8 of the book by Adkins and Weintraub, edited somewhat.

11: Show that the dihedral group of order 8 (denoted D_4 or D_8 depending on convention) and the quaternion group Q_8 have the same character table.

Hint: we determined the irreducible representations of Q_8 in class; they also appear in Example 8.3.8 of Adkins and Weintraub. The irreducible representations of D_8 are determined in Examples 8.1.4(7) and 8.3.7.

16: Recall the definition of the intertwining number of two $\mathbb{C}$-representations V and W of a group G:

$$i(V, W) = \dim_{\mathbb{C}} \operatorname{Hom}_G(V, W).$$

Prove the following lemma stated in class:

Lemma. *Let G be finite, and let V and W be two $\mathbb{C}$-representations of G with characters χ_V and χ_W. Then*

1. *$i(V, W) = i(W, V)$;*
2. *if $V \cong \bigoplus_{i \in I} p_i M_i$ and $W \cong \bigoplus_{i \in I} q_i M_i$ are the decompositions of V and W into irreducibles, then*
$$i(V, W) = \sum_{i \in I} p_i q_i \dim_{\mathbb{C}} \big(\operatorname{End}(M_i) \big);$$
3. *if V is irreducible then $i(V, W)$ is the multiplicity of V in W;*
4. *V is irreducible if and only if $i(V, V) = 1$;*
5. *$i(V, W) = \langle \chi_V, \chi_W \rangle = \dfrac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi}_W(g)$.*

20: Let σ_i be an irreducible $\mathbb{C}$-representation of a finite group G_i for $i = 1, 2$. Prove that $\sigma_1 \otimes \sigma_2$ is an irreducible representation of $G_1 \times G_2$.

22: Compute the "multiplication table" for irreducible complex representations of S_4, i.e., for any two irreducible representations σ_1 and σ_2, decompose σ_1 and σ_2 into a direct sum of irreducibles.

Hint: use the character table for S_4 derived in class.