Math 402 homework for April 3, 2009

These are problems from chapter 7 of the book by Adkins and Weintraub, edited somewhat to be consistent with lecture. As usual, R is always a ring with 1 and all modules are unitary.

2: Let F be a field and let $R = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] : a, b, c \in F \right\}$ be the ring of upper triangular matrices over F. Let $M = F^2$ and make M into a (left) R-module by matrix multiplication. Show that $\mathrm{End}_R(M) \cong F$. Conclude that the converse of Schur's lemma is false, i.e., $\mathrm{End}_R(M)$ can be a division ring without M being a simple R-module.

8: Let M be an R-module of finite length and let K and N be submodules of M. Prove the following length formula:

$$\ell(K + N) + \ell(K \cap N) = \ell(K) + \ell(N).$$

11: Let F be a field, let V be a finite-dimensional vector space over F, and let $T \in \mathrm{End}_F(V)$. We shall say that T is semisimple if the $F[x]$-module V_T is semisimple. If $A \in M_n(F)$, we shall say that A is semisimple if the linear transformation $T_A : F^n \to F^n$ (multiplication by A) is semisimple.

(a) Let $\mathbb{F}_2$ be the field with 2 elements, let $K = \mathbb{F}_2(y)$ be the field of rational functions over $\mathbb{F}_2$ in the indeterminate y, and let $F = K(u)$, where u is a root of $x^2 + y \in K[x]$. Now let A be the companion matrix

$$A = C(x^2 + y) = \begin{bmatrix} 0 & y \\ 1 & 0 \end{bmatrix} \in M_2(K).$$

Show that A is semisimple when considered in $M_2(K)$, but A is not semisimple when considered in $M_2(F)$.

(b) Let L be a subfield of $\mathbb{C}$. Prove that $A \in M_n(L)$ is semisimple if and only if it is also semisimple as a complex matrix.

14: Let R be an integral domain. Then R is semisimple if and only if R is a field.

16: Give an example of a semisimple commutative ring that is not a field.