Math 321 Homework
December 7, 2011 (due December 9)

In the following, the fractional expression $\frac{1}{\infty}$ should always be interpreted as 0.

1. Suppose $1 \le p < q < r \le \infty$.

   (a) Prove that if $f \in \mathcal{C}[a,b]$, then

   $$\|f\|_q \le \|f\|_p^{\lambda} \|f\|_r^{(1-\lambda)}, \quad \text{where } \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

   (b) Show that this also holds for ℓ^p norms.

2. Suppose that $1 \le p < q \le \infty$.

   (a) Prove that for $f \in \mathcal{C}[a,b]$, $\|f\|_p \le (b-a)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$.

   (b) Prove that if (f_n) is a sequence in $\mathcal{C}[a,b]$ and $f_n \to f$ in the L^q norm, then $f_n \to f$ in the L^p norm as well.

   (c) Prove that $\ell^p \subseteq \ell^q$, and that for $a \in \ell^p$, $\|a\|_q \le \|a\|_p$.
   Hint: First consider the case $q = \infty$. For the case $q < \infty$, use the first problem with $r = \infty$.

   (d) Prove that if (a^n) is a sequence of points in ℓ^p (so for each n, $a^n = (a_1^n, a_2^n, \ldots)$ is an ℓ^p sequence) and $a^n \to a$ in the ℓ^p norm, then $a^n \to a$ in the ℓ^q norm as well.