A Quasi-Linear Stochastic Fokker-Planck Equation in σ-Finite Measures*

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Abstract

Solutions of quasi-linear stochastic Fokker-Planck equations for the number density of a system of solute particles in suspension are derived. The initial values and the solutions take values in a class of $\sigma-$finite Borel measures over $\mathbf{R}^d$ where $d \geq 1$. The stochastic driving noise is defined by Itô differentials. For the special case of semi-linear stochastic Fokker-Planck equations the solutions can be represented as solutions of first order stochastic transport equations driven by Stratonovich differentials.

1 Introduction

Kotelenez (2005) considers a microscopic system of N large solute particles suspended in a medium of infinitely many small solvent particles. The state space is $\mathbf{R}^d$ for $d \geq 2$. The equations of motion of the joint system of both large and small particles is governed by infinitely many coupled deterministic nonlinear oscillators. Initial conditions for the positions and velocities of small particles are random. As the average speed of the small particles and a friction coefficient of the large particles tend to infinity, the suitably coarse-grained positions of the large particles are shown to tend towards a system of N correlated Brownian motions. [2]

Since the motion of correlated Brownian particles is random - very similar to independent Brownian motions - the equations for the correlated Brownian motions are called "mesoscopic". To obtain more realistic mesoscopic models for N large particles in suspension, forces acting between the large particles must be modeled as well. The principal steps in the derivation of these extended mesoscopic equations for the positions of the large particles in $\mathbf{R}^d$ are as follows: We first construct solutions of stochastic ordinary differential equations in the sense of Itô (SODE's)[3] such that solutions $r(t, \tilde{\mathcal{Y}}, q)$ depend "nicely" on some measure valued process $\tilde{\mathcal{Y}}$ and on the initial condition q. To proceed from SODE's to stochastic partial differential equations (SPDE's) for the mass distribution of the large particles (or "number density"),

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[2]Kotelenez, Leitman and Mann (2007) investigate the qualitative behavior of a system of correlated Brownian motions. They show that for dimension $d \geq 2$ the long-time behavior is the same as the long-time behavior of independent Brownian motions. For short times, however, correlated Brownian motions show attractive behavior when the large particles are close to each other. This last property is consistent with the depletion phenomenon of colloids.

[3]Cf. the following (2.7) and (2.8).

governed by the *flow* $q \mapsto r(t, \tilde{\mathcal{Y}}, q)$ we proceed in a second step as follows: Suppose $r(t, \tilde{\mathcal{Y}}, q)$ is measurable in q with respect to the Borel $\sigma-$algebra $\mathcal{B}^d$ of $\mathbf{R}^d$ and integrable with respect to some initial measure μ_0. If $\mu_0(\mathbf{R}^d) < \infty$, we define a finite measure- valued process[4]

$$\mathcal{Y}(t)(dr) := \int \delta_{(r(t,\tilde{\mathcal{Y}},q))}(dr)\mu_0(dq) \tag{1.1}$$

and obtain $\mathcal{Y}(t, \mathbf{R}^) = \mu_0(\mathbf{R}^)$. The process $\mathcal{Y}(t)$ is the solution of a bilinear SPDE[5] and is called the empirical (measure) process of the large particles. If, in addition to the previous step, the input process $\tilde{\mathcal{Y}}(\cdot)$ equals $\mathcal{Y}(\cdot)$ from the left hand side of (1.1), then $\mathcal{Y}(t)$ is the solution of a semi-linear or quasi-linear SPDE.[6] $\mathcal{X}(t)$ instead of $\mathcal{Y}(t)$.

If the initial distribution is in the space of finite Borel measures on $\mathbf{R}^d$ Kotelenez (1995a,b) extends the empirical processes associated with (2.7) and (2.8), respectively, by continuity from discrete initial distributions to general initial distribution. Cf. also Kurtz and Xiong (1999), Dawson, Vaillancourt and Wang (2000) as well as Dorogovtsev (2004) for related work and some generalizations. This *extension by continuity* is one of the main tools in the derivation of SPDE's from the empirical measure processes of finitely many SODE's. Our goal is to derive a solution of (4.2) for initial distributions in a space of $\sigma-$finite measures. Solutions of SPDE's in the space of $\sigma-$finite measures can define shift and rotation invariant random fields (cf. Kotelenez 2007a).

The paper is organized as follows: In Section 2 we introduce SODE's for the positions of N (large) particles and the necessary notation. For the proofs of the theorems on SODE's we refer to Kotelenez (2007a). In Section 3 we introduce a class of stochastic flows which leave a class of $\sigma-$finite initial conditions invariant. It follows that the solutions of the SODE's from Section 2 define such stochastic flows. In Section 4 we introduce the stochastic partial differential equations (SPDE's), associated with the SODE's from Section 2. Employing the results from Section 3 we show that the quasi-linear SPDE (4.2) has a solution in $\sigma-$finite measures which are characterized by a certain weight function. For the special case of semi-linear SPDE's the results of Kotelenez (2007b) imply that the solutions may be represented as solutions of first order stochastic transport equations driven by Stratonovich differentials. Section 5 is an appendix and contains some simple results on the weight function.

2 Stochastic Ordinary Differential Equations

Define for $d \geq 1$ the following metric on $\mathbf{R}^d$

$$\rho(r - q) := |r - q| \wedge 1, \tag{2.1}$$

where $r, q \in \mathbf{R}^d$, $|r - q|$ is the Euclidean distance on $\mathbf{R}^d$ and "$\wedge$" denotes "minimum". The particle distributions will be measures. Set

[4](1.1) is by definition the image of the initial measure μ_0 under the "flow" $q \mapsto r(t, \tilde{\mathcal{Y}}, q), i.e.,$ $\int_A \mathcal{Y}(t)(dr) = \int \delta_{(r(t,\tilde{\mathcal{Y}},q))}(A)\mu_0(dq) = \mu_0(r_\varepsilon^{-1}(t, \tilde{\mathcal{Y}}, s, q))(A)$, where $A \in \mathcal{B}^d$. Here and in what follows we do not indicate the domain of integration when integrating over $\mathbf{R}^d$

[5]Cf. Section 4, (4.1).

[6]Cf. Section 4, (4.2). For this case we use the notation

$$\mathbf{M}_f := \{\mu : \mu \text{ is a finite Borel measure on } \mathbf{R}^d\}. \tag{2.2}$$

Let γ_f be the Wasserstein distance on $\mathbf{M}_f$. For the precise definition we refer to Section 5, (5.4). We note that $(\mathbf{M}_f, \gamma_f)$ is a complete and separable metric space (cf. Kotelenez (2007a). Further, we note that γ_f is the restriction of a norm [7] on a space of distributions to the cone of nonnegative finite Borel measures and that

$$\gamma_f(\mu) = \mu(\mathbf{R}^d). \tag{2.3}$$

By $\mathbf{M}_\infty$ we denote the $\sigma-$ finite Borel measures μ on $\mathbf{R}^d$, i.e., μ is $\sigma-$ finite. For the description of those $\mu \in \mathbf{M}_\infty$, which have a density with respect to the Lebesgue measure, we choose the following standard weight function making the constants integrable over $\mathbf{R}^d$ (cf. Section 5):

$$\varpi(r) = (1 + |r|^2)^{-\gamma}, \quad \gamma > \tfrac{d}{2}. \tag{2.4}$$

In what follows we write $\varpi\mu$ etc. to denote the finite measure which is represented as μ with density ϖ. Set

$$\mathbf{M}_{\infty,\varpi} := \{\mu \in \mathbf{M}_\infty : \textstyle\int \varpi(q)\mu(dq) < \infty\} \tag{2.5}$$

where the integration is taken over $\mathbf{R}^d$. If we define for $\mu, \nu \in \mathbf{M}_{\infty,\varpi}$

$$\gamma_\varpi(\mu - \nu) := \gamma_f(\varpi(\mu - \nu))$$

we obtain that $(\mathbf{M}_{\infty,\varpi}, \gamma_\varpi)$ is isometrically isomorphic to $(\mathbf{M}_f, \gamma_f)$. Therefore, $(\mathbf{M}_{\infty,\varpi}, \gamma_\varpi)$ is also a complete separable metric space (and even a separable Fréchet space). For the remainder of this section let $(\mathbf{M}, \gamma) \in \{(\mathbf{M}_f, \gamma_f), (\mathbf{M}_{\infty,\varpi}, \gamma_\varpi)\}$.

The stochastic set-up is provided as follows: $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω and be $\mathcal{F}_t$-adapted (including all initial conditions in stochastic ordinary differential equations (SODE's) and stochastic partial differential equations (SPDE's)). Moreover, the processes are assumed to be $dP \otimes dt$-measurable, where dt is the Lebesgue measure on $[0, \infty)$. Let $w_\ell(dr, dt)$ be i.i.d. real valued space-time white noises on $\mathbf{R}^d \times \mathbf{R}_+$, $\ell = 1, \ldots, d$ (cf. Definition 2.2 in Kotelenez (2007a)). Set $w(p,t) := (w_1(p,t), \ldots, w_d(p,t))^T$, where "$T$" denotes the transpose.

Notation

(i) $C([s,T]; \mathbf{M})$ is the space of continuous $\mathbf{M}$-valued functions, defined on $[s,T]$. $L_{0,\mathcal{F}_s}(\mathbf{M})$ is the space of $\mathbf{M}$-valued $\mathcal{F}_s$-measurable random variables, and $L_{0,\mathcal{F}}(C([s,T]; \mathbf{M}))$ is the space of $\mathbf{M}$-valued adapted and $dt \otimes dP$-measurable processes with sample paths in $C([s,T]; \mathbf{M})$ a.s. Further, let $p \geq 1$. $L_{p,\mathcal{F}_s}(\mathbf{M})$ and $L_{p,\mathcal{F}}(C([s,T]; \mathbf{M}))$ are those elements in $L_{0,\mathcal{F}_s}(\mathbf{M})$ and $L_{0,\mathcal{F}}(C([s,T]; \mathbf{M}))$, respectively, with finite $p-$th moments of $\gamma(\mu)$, where for the processes $\sup_{s \leq t \leq T} \gamma^p(\mu(t))$ has to be integrable. For $p \in \{0\} \cup [1, \infty)$ $L_{p,\mathcal{F}}(C([s,\infty); \mathbf{M}))$ is the space of those processes $\mu(\cdot)$ such that $\mu(\cdot \wedge T) \in L_{p,\mathcal{F}}(C([s,T]; \mathbf{M}))$ $\forall T > s$.

(ii) Let τ_n be a sequence of localizing stopping times for $mu_\ell(\cdot, \omega)$ and set for $p \geq 1$

[7] It follows that $(\mathbf{M}_f, \gamma_f)$ is a separable Fréchet space.

$L_{loc,p,\mathcal{F}}(C([s,T];\mathbf{M}))$

$:= \{\mu(\cdot) \in \mathrm{L}_{0,\mathcal{F}}(C([s,T];\mathbf{M})) : \quad \exists$ "l.s.t." $(\tau_n := \tau_n(\mu))$ s.t. $\mathrm{E}(\sup_{s\le t\le T\wedge\tau_n}\gamma^p(\mu(t))) < \infty\} \quad \forall n,$
$L_{loc,p,\mathcal{F}}(C((s,T];\mathbf{M}))$
$:= \{\mu(\cdot) \in \mathcal{M}_{[s,T]} : \quad \exists$ "l.s.t." $(\tau_n := \tau_n(\mu))$ s.t. $E(\sup_{s\le t\le T\wedge\tau_n}\gamma^p($
$mu(t))1_{\{\tau_n>s\}}) < \infty\} \quad \forall n,$
$L_{loc,p,\mathcal{F}_s}(\mathbf{M}) := \{\mu \in L_{0,\mathcal{F}_s}(\mathbf{M}) :)) : \quad \exists$ "l.s.t." $(\tau_n := \tau_n(\eta))$ s.t. $\mu 1_{\tau_n>s} \in L_{p,\mathcal{F}_s}(\mathbf{M})\} \quad \forall n \}$

where "l.s.t." means "localizing stopping time". If the particular level n at which we stop or truncate the random maps is irrelevant (as it will be most of the time), we drop the subscript n and call τ a "localizing" stopping time. This means that for each level n there is such a τ with the properties, described above. ■

Let $\mathcal{M}_{d\times d}$ denote the $d\times d$-matrices over $\mathbf{R}$. We now define the coefficients for the stochastic ordinary differential equations (SODE's):

$$F : \mathbf{R}^d \times \mathbf{M} \times \mathbf{R} \to \mathbf{R}^d; \quad \mathcal{J} : \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{M} \times \mathbf{R} \to \mathcal{M}_{d\times d}. \tag{2.6}$$

We consider two types of stochastic ordinary differential equations:

$$\left.\begin{array}{rl} dr^i(t) = & F(r^i(t),\tilde{\mathcal{Y}}(t),t)dt + \int \mathcal{J}(r^i(t),p,\tilde{\mathcal{Y}}(t),t)w(dp,dt) \\ r^i(s) = & r^i_s \in L_{2,\mathcal{F}_s}(\mathbf{R}^d), \quad \tilde{\mathcal{Y}} \in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M})), \quad i=1,...,N, \end{array}\right\} \tag{2.7}$$

where $L_{2,\mathcal{F}_s}(\mathbf{R}^d)$ is the space of $\mathbf{R}^d$-valued, $\mathcal{F}_s$-measurable random variables r_s such that $E|r_s|^2 < \infty$. (2.7) describes the motion of a system of diffusing particles in a random environment (represented by $\tilde{\mathcal{Y}},\ w_\ell,\ \ell = 1,...,d,\ \beta^{\perp,i},\ i = 1,...,N.$). The $\mathbf{R}^{Nd}-$valued process with $d-$dimensional components $r^i(t)$ will be denoted $r_N(t)$.

$$\left.\begin{array}{rl} dr^i_N(t) = & F(r^i_N(t),\mathcal{X}_N(t),t)dt + \int \mathcal{J}(r^i_N(t),p,\mathcal{X}_N(t),t)w(dp,dt), \\ r^i_N(s) = & r^i_s \in L_{2,\mathcal{F}_s}(\mathbf{R}^d),\ i=1,\ldots N,\ \mathcal{X}_N(t) := \sum_{i=1}^N m_i\delta_{r^i_N(t)}, \quad m_i \ge 0. \end{array}\right\} \tag{2.8}$$

(2.8) represents the motion of N interacting and diffusing particles in the random environment(represented by $w_\ell, \ell = 1,...,d, \beta^{\perp,i}, i = 1,...,N$). δ_{r^i} is the point measure concentrated in r^i. $\mathcal{X}_N(t)$ is the *empirical measure process* associated with (2.8). The general measure process $\tilde{\mathcal{Y}}(t)$ in (2.7) has been replaced by the empirical measure process of the solution of (2.8). Therefore, the $N\cdot d$-vector valued solution of (2.8) depends on N, whereas in (2.7) it is independent of N.

Suppose $(r_\ell,\mu_\ell,t) \in \mathbf{R}^d\times\mathbf{M}\times\mathbf{R},\ \ell = 1,2$. Let $c_F,\ c_{\mathcal{J},\sigma^\perp},\ c_{F,\mathcal{J},\sigma^\perp} \in (0,\infty)$. Assume global Lipschitz and boundedness conditions:

$$\left.\begin{array}{l} (a) \quad |F(r_1,\mu_1,t) - F(r_2,\mu_2,t)| \le c_F\{(\gamma(\mu_1)\vee\gamma(\mu_2))\rho(r_1-r_2) + \gamma(\mu_1-\mu_2)\}, \\ \quad \sum_{k,\ell=1}^d [\int(\mathcal{J}_{k\ell}(r_1,p,\mu_1,t) - \mathcal{J}_{k\ell}(r_2,p,\mu_2,t))^2 dp \\ \quad \le c^2_{\mathcal{J}}\{(\gamma^2(\mu_1)\vee\gamma^2(\mu_2))\rho^2(r_1-r_2) + \gamma^2(\mu_1-\mu_2)\}; \\ \\ (b) \quad |F_\varepsilon(r,\mu,t)|^2 + \sum_{k,\ell=1}^d \{\int \mathcal{J}^2_{\varepsilon,k\ell}(r,p,\mu,t)dp\} \le c_{F,\mathcal{J}}. \end{array}\right\} \tag{2.9}$$

The constants $c_F, c_{\mathcal{J}}$ and $c_{F,\mathcal{J}}$ in (2.9) may also depend on the space dimension d.

In Section 4.4 of Kotelenez (2007a) a number of examples of coefficients $F, \mathcal{J}$ satisfying Hypothesis 4.1 is provided.

Denote a solution of (2.7), if it exists, by $r_N(t, \tilde{\mathcal{Y}}, r_{N,s}, s)$. Further, if f is a stochastic process on $[s, \infty)$ with values in some metric space, we set for $t \geq s$

$$(\pi_{s,t} f)(u) := f(u \wedge t), (u \geq s).$$

Further, let $\mathcal{G}_{s,t}$ (resp. $\mathcal{G}_t$) be the σ-algebra generated by $w(dp, du)$ between s and t (resp. 0 and t) for $t \geq s$. The cylinder set filtration on $C([s, \infty); \mathbf{M})$ is denoted $\mathcal{F}_{\mathbf{M},s,t}$. We write $\mathcal{F}_{\mathbf{M},t}$ if $s = 0$. Completed $\sigma-$algebras will be denoted with a bar on top of the σ-algebra, e.g. $\bar{\mathcal{G}}_{s,t}$.

The proofs of the following two theorems can be found in Kotelenez (2007a), Section 4, and alternative versions in Kotelenez (1995a,b).

Theorem 2.1

Assume (2.9) Then:

1) To each $s \geq 0$, $r_s^k \in L_{2,\mathcal{F}_s}(\mathbf{R}^d)$, $\tilde{\mathcal{Y}} \in L_{loc,2,\mathcal{F}}(C((s,T]; \mathbf{M}))$ (2.7) has a unique solution $r^k(\cdot, \tilde{\mathcal{Y}}, r_s^k, s) \in L_{loc,2,\mathcal{F}}(C([s,T]; \mathbf{R}^d))$.

2) Let $\tilde{\mathcal{Y}}_i \in L_{loc,2,\mathcal{F}}(C((s,T]; \mathbf{M}))$ and $r_{s,i}^k \in L_{2,\mathcal{F}_s}(\mathbf{R}^d)$, $i = 1, 2$. Then for any $T \geq s$ and any stopping time $\tau \geq s$, which is localizing for $\tilde{\mathcal{Y}}_i$, i = 1,2,

$$\left.\begin{array}{rl} & E \sup_{s \leq t \leq T\wedge\tau} \tilde{\rho}^2(r^k(t, \tilde{\mathcal{Y}}_1, r_{s,1}^k, s) - r^k(t, \tilde{\mathcal{Y}}_2, r_{s,2}^k, s))1_{\{\tau > s\}} \\ \leq & c_{T,F,\mathcal{J},\sigma,\tilde{\mathcal{Y}},\tau}\{E(\tilde{\rho}^2(r_{s,1}^k - r_{s,2}^k)1_{\{\tau>s\}}) + E\int_s^{T\wedge\tau}(\gamma^2(\tilde{\mathcal{Y}}_1(u) - \tilde{\mathcal{Y}}_2(u))1_{\{\tau>s\}}du\}. \end{array}\right\} \tag{2.10}$$

Further, with probability 1 uniformly in $t \in [s, \infty)$

$$r^k(t, \tilde{\mathcal{Y}}, r_{s,1}^k, s) \equiv r^k(t, \pi_{s,t}\tilde{\mathcal{Y}}, r_{s,1}^k, s). \tag{2.11}$$

3) For any $N \in \mathbf{N}$ there is a $\mathbf{R}^{dN}$-valued map in the variables $(t, \omega, \mu(\cdot), r_N, s)$, $0 \leq s \leq t < \infty$ such that for any fixed $s \geq 0$

$$\overline{r}_N(\cdot, \ldots, \cdot, s) : \Omega \times C([s,T]; \mathbf{M}) \times \mathbf{R}^{dN} \to C([s,T]; \mathbf{R}^{dN}),$$

and the following holds:

(i) For any $t \in [s,T]$ $\overline{r}_N(t, \cdot, \ldots, \cdot)$ is $\overline{\mathcal{G}}_{s,t} \otimes \mathcal{F}_{\mathbf{M},s,t} \otimes \mathcal{B}^{dN} \otimes \mathcal{B}_{[0,t]} - \mathcal{B}^{dN}$-measurable.

(ii) The i-th d-vector of $\overline{r}_N = (\overline{r}^1, \ldots, \overline{r}^i, \ldots \overline{r}^N)$ depends only on the i-th d-vector initial condition $r_s^i \in L_{2,\mathcal{F}_s}(\mathbf{R}^d)$ and the Brownian motion $\beta^{\perp,i}(\cdot)$, in addition to its dependence on $w(dq, dt)$ and $\tilde{\mathcal{Y}} \in L_{2,\mathcal{F}}(C([s,T]; \mathbf{M}))$, and with probability 1 (uniformly in $t \in [s, \infty)$)

$$\overline{r}^i_N(t,\cdot,\tilde{\mathcal{Y}},r^i_s,s) \equiv r^i(t,\tilde{\mathcal{Y}},r^i_s,s), \tag{2.12}$$

where the right hand side of (2.12) is the ith d-dimensional component of the solution of (2.7).

(iii) If $u \geq s$ is fixed, then with probability 1 (uniformly in $t \in [u,\infty)$)

$$\overline{r}_N(t,\cdot,\pi_{u,t}\tilde{\mathcal{Y}},\overline{r}_N(u,\cdot,\pi_{s,u}\tilde{\mathcal{Y}},r_{N,s},s),u) \equiv \overline{r}_N(t,\cdot,\pi_{s,t}\tilde{\mathcal{Y}},r_{N,s},s). \tag{2.13}$$

■

Next, we consider the $\mathbf{R}^{dN}$-valued system of coupled SODE's (2.8). Since for each ω the initial measure is a finite sum of point measures, it is finite. Therefore,

$$\mathcal{X}_N(s) := \sum_{i=1}^N m_i\delta_{r^i(s)}, \in L_{0,\mathcal{F}_s}(\mathbf{M}).$$

Further, a solution of (2.8), if it exists, preserves the initial mass, i.e., $\mathcal{X}_N(\cdot,\mathbf{R}^d) \equiv \sum_{i=1}^N m_i$, where $\mathcal{X}_N(t) := \sum_{i=1}^N m_i\delta_{r^i(t)}$. Therefore, we may take $\tilde{\mathcal{Y}}(t) := \mathcal{X}_N(t) := \sum_{i=1}^N m_i\delta_{r^i(t)}$ in Theorem 2.1.

Theorem 2.2

Assume (2.9) in addition to $\mathcal{X}_N(s) \in L_{0,\mathcal{F}_s}(\mathbf{M}_f)$. Then, to each initial condition $r_N(s) \in L_{0,\mathcal{F}_s}(\mathbf{R}^{dN})$ (2.8) has a unique solution $r_N(\cdot,r_N(s)) \in L_{0,\mathcal{F}}(C([s,\infty);\mathbf{R}^{dN}))$ which is a Markov process on $\mathbf{R}^{dN}$. ■

3 Stochastic Flows

Let us now return to our approach to mesoscopic equations. Equation (1.1) implies *mass conservation*, i.e., $\mathcal{Y}(t,\omega,\mathbf{R}^d) = \mu_0(\omega,\mathbf{R}^d)$. Hence, we have in this case $\gamma_f(\mathcal{Y}(t,\omega)) = \mathcal{Y}(t,\omega,\mathbf{R}^d) = \mu_0(\omega,\mathbf{R}^d) = \gamma_f(\mu_0(\omega)) < \infty$. Therefore, bounds on the initial measure also yield bounds for the measure uniformly in $t \geq 0$, if we work with $\mathbf{M}_f$-valued initial conditions. The situation with $\mathbf{M}_{\infty,\varpi}-$valued initial conditions is not so straightforward. Although we might think of $\int_A \varpi(q)\mu(dq)$ as a finite measure on the Borel sets even if $\mu(\mathbf{R}^d) = \infty$, we do not have in general $\int \varpi(q)\mathcal{Y}(t,dq) = \int \varpi(q)\mu_0(dq)$ - even if $\mu_0(\mathbf{R}^d) = \mathcal{Y}(t,\mathbf{R}^d) < \infty$. The problem is that the weight function ϖ "distorts" the mass distribution. However, if we assume $E\gamma_{\infty,\varpi}(\mu_0) < \infty$, we can find an important subspace of $L_{0,\mathcal{F}}(C([s,\infty);\mathbf{M}_{\infty,\varpi}))$ where this assumption implies $E\sup_{0\leq t\leq T}\gamma_{\infty,\varpi}(\mu(t)) < \infty$.

Definition 3.1

Ψ is the set of adapted measurable flows φ from $[0,T]\times\mathbf{R}^d$ into $\mathbf{R}^d$ which satisfy the following conditions:

(i) $\varphi(0,q) \equiv q$.

(ii) $t \longmapsto \varphi(t,q)$ is continuous a.s. $\forall q$ and measurable in $q \in \mathbf{R}^d$.

(iii) $\forall q \;\; \varphi(t,q)$ is a square integrable semi-martingale. If $b(t,q)$ and $m(t,q)$ are the process of bounded variation and the martingale in the Doob-Meyer decomposition, respectively, both $b(t,q)$ and the mutual quadratic variations of the one dimensional martingale components, $[m_k(t,q), m_\ell(t,q)]$ are differentiable with respect to t and the derivatives are bounded uniformly in $\omega \in \Omega, q \in \mathbf{R}^d$ and $t \in [0,T]$ for all $T > 0$. ■

Clearly, condition (iii) implies for $\varphi \in \Psi$

$$\sup_q E \sup_{0 \le s \le T} |\varphi(s,q) - q|^2 < \infty.$$

Proposition 3.2

(a) Suppose $\mu_0 \in \mathcal{M}_{\infty,\varpi}$ and $\varphi \in \Psi$. Then

$$t \longmapsto \mu(\cdot) := \mu_0 \circ \varphi^{-1}(\cdot) \in L_{0,\mathcal{F}}(C([0,\infty); \mathbf{M}_{\infty,\varpi})). \tag{3.1}$$

(b) Suppose, in addition, that for $p \ge 1 \;\; E\gamma_\varpi^p(\mu_0) < \infty$. Then

$$E \sup_{0 \le t \le T} \gamma_\varpi^p(\mu(t)) \le c_T E\gamma_\varpi^p(\mu_0) < \infty \tag{3.2}$$

where the finite constant c_T depends only on T and the bounds of the derivatives of the characteristics of the semi-martingale.

Proof

(i) The definition of $\mu(\cdot)$ is equivalent to

$$\mu(t,dr) = \int \delta_{\varphi(t,q)}(dr)\mu_0(dq).$$

So,

$$\gamma_\varpi(\mu(t)) \le \int \varpi(r)\mu(t,dr) = \int \varpi(\varphi(t,q))\mu_0(dq).$$

Let $\beta > 0$ and consider $\varpi_\beta(r) := \varpi^\beta(r)$. By Itô's formula $\forall q$

$$\begin{aligned}&\varpi_\beta(\varphi(t,q))\\ &= \varpi_\beta(q) + \textstyle\int_0^t (\nabla \varpi_\beta)(\varphi(s,q)) \bullet [db(s,q) + dm(s,q)] + \frac{1}{2}\sum_{k,\ell=1}^d (\partial^2_{k\ell}\varpi_\beta)(\varphi(s,q))d[m_k(s,q), m_\ell(s,q)].\end{aligned}$$

Hence, by the Burkholder-Davies-Gundy inequality and the usual inequality for increasing processes,

$$\left.\begin{array}{l}
E\sup_{0\le t\le T}\varpi_\beta(\varphi(t,q)\\
\le \varpi_\beta(q)+c\int_0^T\sum_{k=1}^d E|\partial_k\varpi_\beta(\varphi(s,q))|d|b_k(s,q)|\\
+cE\sqrt{\int_0^T|\sum_{k,\ell}^d\partial_k\varpi_\beta(\varphi(s,q))\partial_\ell\varpi_\beta(\varphi(s,q))|d[m_k(s,q),m_\ell(s,q)]}\\
+cE\int_0^T\sum_{k,\ell}^d|\partial^2_{k,\ell}\varpi_\beta(\varphi(s,q))|d[m_k(s,q),m_\ell(s,q)]\\
\le \varpi_\beta(q)+c\int_0^T\sum_{k=1}^d E\varpi_\beta(\varphi(s,q))d|b_k(s,q)|\\
+cE\sqrt{\int_0^T\sum_{k,\ell}^d\varpi_\beta^2(\varphi(s,q))d\{[m_k(s,q)]+[m_\ell(s,q)]\}}\\
+cE\int_0^T\sum_{k,\ell}^d\varpi_\beta(\varphi(s,q))d\{[m_k(s,q)]+[m_\ell(s,q)]\}\\
\text{(by (5.1))}\\
\le \varpi_\beta(q)+c\int_0^T\sum_{k=1}^d E\varpi_\beta(\varphi(s,q))d|b_k(s,q)|\\
+c\sqrt{\int_0^T\sum_{k,\ell}^d E\varpi_\beta^2(\varphi(s,q))d\{[m_k(s,q)]+[m_\ell(s,q)]\}}\\
+cE\int_0^T\sum_{k,\ell}^d\varpi_\beta(\varphi(s,q))d\{[m_k(s,q)]+[m_\ell(s,q)]\}
\end{array}\right\}\qquad(3.3)$$

From Itô's formula and (5.1) we obtain for $\tilde\beta>0$

$$\begin{array}{l}
E\varpi_{\tilde\beta}(\varphi(t,q)\\
\le \varpi_{\tilde\beta}(q)+c\int_0^T\sum_{k=1}^d E\varpi_{\tilde\beta}(\varphi(s,q))d|b_k(s,q)|\\
+cE\int_0^T\sum_{k,\ell}^d\varpi^2_{\tilde\beta}(\varphi(s,q))d\{[m_k(s,q)]+[m_\ell(s,q)]\}.
\end{array}$$

Gronwall's inequality implies

$$E\varpi_{\tilde\beta}(\varphi(t,q)\le c_{T,b,[m]}\varpi_{\tilde\beta}(q),\qquad(3.4)$$

where the finite constant $c_{T,b,[m]}$ depends on $T,\tilde\beta$, the bounds of the derivatives of b and $\{[m_k(s,q)]+[m_\ell(s,q)]\}$. We employ (3.4) (with $\tilde\beta:=2\beta$) to estimate the square root in the last inequality of (3.3):

$$c\sqrt{\int_0^T\sum_{k,\ell}^d E\varpi_\beta^2(s,q)d\{[m_k(s,q)]+[m_\ell(s,q)]\}}\le c\sqrt{\int_0^T\sum_{k,\ell}^d E\varpi_\beta^2(s,q)c_{T,b,[m]}ds]\}}\le\hat c_{T,b,[m]}\varpi_\beta(q).\qquad(3.5)$$

To estimate the remaining integrals in (3.3) we again use Gronwall's inequality and obtain altogether

$$E\sup_{0\le t\le T}\varpi_\beta(\varphi(t,q))\le\bar c_{T,b,[m]}\varpi_\beta(q),\qquad(3.6)$$

where $\bar c_{T,b,[m]}$ depends only upon T, β and upon the bounds of the characteristics of φ, but not upon φ itself. In what follows we choose $\beta=1$.

We may change the initial measure μ_0 on $\mathcal{F}_0-$measurable set of arbitrary small probability such that

$$ess\sup_\omega\int\varpi(q)\mu_0(dq)<\infty.$$

Therefore, $E\int\varpi(q)\mu_0(dq)<\infty$. We may use conditional expectations instead of absolute ones in the preceding calculations[8] and assume in the first step that μ_0 is deterministic. Hence $E\sup_{0\le t\le T}\varpi(\varphi(t,q))$ is integrable against the initial measure μ_0, and we obtain

[8] Cf. (3.3).

$$E \sup_{0\le t\le T}\int \varpi(\varphi(t,q))\mu_0(dq) \le \int E\sup_{0\le t\le T}\varpi(\varphi(t,q))\mu_0(dq) \le c_{T,b}E\int \varpi(q)\mu_0(dq) < \infty. \tag{3.7}$$

As a result we have a.s.

$$\sup_{0\le t\le T}\gamma_\varpi(\mu(t)) \le \sup_{0\le t\le T}\int \varpi(\varphi(t,q))\mu_0(dq) \le \int \sup_{0\le t\le T}\varpi(\varphi(t,q))\mu_0(dq) < \infty. \tag{3.8}$$

(ii) We show that $\mu(\cdot)$ is continuous (a.s.) in $\mathbf{M}_{\infty,\varpi}$. Let $t > s \ge 0$. Then,

$$\begin{aligned}\gamma_\varpi(\mu(t)-\mu(s)) &= \sup_{\|f\|_{L,\infty}\le 1}|\textstyle\int [f(\varphi(t,q))\varpi(\varphi(t,q)) - f(\varphi(s,q))\varpi(\varphi(s,q))]\mu_0(dq)|\\ &\le \sup_{\|f\|_{L,\infty}\le 1}\textstyle\int |f(\varphi(t,q)) - f(\varphi(s,q))|\varpi(\varphi(t,q))]\mu_0(dq)\\ &+\sup_{\|f\|_{L,\infty}\le 1}|\textstyle\int f(\varphi(s,q))[\varpi(\varphi(t,q)) - \varpi(\varphi(s,q))]\mu_0(dq)|\\ &=: I(t,s) + II(t,s).\end{aligned}$$

Since the Lipschitz constant for all f in the supremum is 1

$$I(t,s) \le \textstyle\int \rho(\varphi(t,q)-\varphi(s,q))\varpi(\varphi(t,q))\mu_0(dq).$$

Further, $\varphi(\cdot,q)$ is uniformly continuous in $[0,T]$ and $\rho(\varphi(t,q)-\varphi(s,q)) \le 1$. Therefore, we conclude by (3.8) and Lebesgue's dominated convergence theorem that a.s.

$$\lim_{\delta\to 0}\sup_{0\le s<t\le T,|t-s|\le\delta} I(t,s) = 0.$$

Similarly,

$$|f(\varphi(s,q)(\varpi(\varphi(t,q)) - \varpi(\varphi(s,q)))| \le 2\sup_{0\le t\le T}\varpi(\varphi(t,q)),$$

where the right hand side is integrable with respect to $\mu_0(dq)$ a.s. by (3.8). Therefore,

$$\begin{aligned}&II(t,s) = \sup_{\|f\|_{L,\infty}\le 1}|\textstyle\int f(\varphi(s,q))[\varpi(\varphi(t,q)) - \varpi(\varphi(s,q))]\mu_0(dq)|\\ &\le \textstyle\int |\varpi(\varphi(t,q)) - \varpi(\varphi(s,q))|\mu_0(dq) \longrightarrow 0, \text{ as } |t-s|\to 0\end{aligned}$$

by the continuity of the integrand in t and by Lebesgue's dominated convergence theorem. Thus, we have shown that with probability 1

$$\lim_{\delta\to 0}\sup_{0\le s<t\le T,|t-s|\le\delta}\gamma_\varpi(\mu(t)-\mu(s)) = 0, \tag{3.9}$$

i.e., $\mu(\cdot)$ has continuous sample paths in $\mathbf{M}_{\infty,\varpi}$.

(iii) The adaptedness of $\mu(\cdot)$ follows directly from the construction. This finishes the proof of Part (a).

To show Part (b) we may again assume that the initial distribution is deterministic. The proof for $p = 1$ follows immediately from (3.6). We may, therefore, without loss of generality, assume $p > 1$. Hence, we obtain

$$
\begin{gathered}
E\sup_{0\le t\le T}\gamma_{\varpi}^{p}(\mu(t))\le E[\int\sup_{0\le t\le T}\varpi(\varphi(t,q))\mu_0(dq)]^p\\
=E[\int\sup_{0\le t\le T}\varpi(\varphi(t,q))\varpi^{-\frac{p-1}{p}}(q)\varpi^{\frac{p-1}{p}}(q)\mu_0(dq)]^p\\
\le E[\int\sup_{0\le t\le T}\varpi^p(\varphi(t,q))\varpi^{-(p-1)}(q)\mu_0(dq)][\int\varpi(q)\mu_0(dq)]^{p-1}\\
\text{(by Hölder's inequality)}\\
=[\int E\{\sup_{0\le t\le T}\varpi^p(\varphi(t,q))\}\varpi^{-(p-1)}(q)\mu_0(dq)][\int\varpi(q)\mu_0(dq)]^{p-1}\\
\text{(by Fubini's theorem)}\\
\le c_T[\int\varpi(q)^p\varpi^{-p+1}\mu_0(dq)][\int\varpi(q)\mu_0(dq)]^{p-1}\\
\text{(by (3.6) for }\beta=p\text{)}\\
=c_T[\int\varpi(q)\mu_0(dq)]^p\\
=c_T\gamma_{\varpi}^{p}(\mu_0)\quad\text{(by the definition of }\gamma_{\varpi}\text{).}
\end{gathered}
$$

■

Remark 3.3

By Part 3) of Theorem 2.1 and the boundedness of the coefficients F,$\mathcal{J}$ and σ_i the map

$$q\mapsto\psi^i(t,q):=\bar{r}_N^i(t,\tilde{\mathcal{Y}},q) \tag{3.10}$$

is in Ψ from Proposition 3.2 for arbitrary input process $\tilde{\mathcal{Y}}$ and $i=1,...,N$. ■

4 Stochastic Partial Differential Equations - Infinite Mass

Employing the linearity of the duality $<\cdot,\cdot>$ between measures and functions and the fact that the test functions from $C_c^3(\mathbf{R}^d,\mathbf{R})$ uniquely determine the measure, Itô's formula implies that the empirical process associated with (2.7), $\mathcal{Y}_N(t)$, is a *weak solution* of the bilinear SPDE

$$d\mathcal{Y}=(\tfrac{1}{2}\sum_{k,\ell=1}^{d}\partial_{k\ell}^2(D_{k\ell}(\tilde{\mathcal{Y}},\cdot,t)\mathcal{Y})-\bigtriangledown\bullet(\mathcal{Y}F(\cdot,\tilde{\mathcal{Y}},t))dt-\bigtriangledown\bullet(\mathcal{Y}\int\mathcal{J}(\cdot,p,\tilde{\mathcal{Y}},t)w(dp,dt)) \tag{4.1}$$

with initial condition $\mathcal{Y}(s)=\sum_{i=1}^{N}m_i\delta_{r_s^i}$ (cf. Kotelenez (2007a), Section 8. Note that we use "weak solution" as is customary in the area of partial differential equations. Since we do not change the probability space and work strictly with the same stochastic driving term, our solution is a "*strong (Itô) solution*" in the terminology of stochastic analysis.

The empirical process associated with (2.8) is $\mathcal{X}_N(t)$, which is explicitly defined in (2.8). We verify that $\mathcal{X}_N\in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M}_f))$. Replacing $\tilde{\mathcal{Y}}$ by $\mathcal{X}_N$ in (4.1), we also obtain that the empirical process associated with (4.10), $\mathcal{X}_N(t)$, is a weak solution of the quasi-linear SPDE:

$$d\mathcal{X}=(\tfrac{1}{2}\sum_{k,\ell=1}^{d}\partial_{k\ell,r}^2(D_{k\ell}(\mathcal{X},\cdot,t)\mathcal{X})-\bigtriangledown\bullet(\mathcal{X}F(\cdot,\mathcal{X},t)))dt-\bigtriangledown\bullet(\mathcal{X}\int\mathcal{J}(\cdot,p,\mathcal{X},t)w(dp,dt)) \tag{4.2}$$

with initial condition $\mathcal{X}(s)=\sum_{i=1}^{N}m_i\delta_{r_s^i}$.

Remark 4.1

(4.2) has the form of a deterministic Fokker-Planck equation (cf. Arnold (1973)) perturbed by a stochastic term. The generalized density, described by (4.2) (or (4.1)) corresponds to what is called "*number density*" in physics and engineering. Therefore, we call (4.2) a stochastic Fokker-Planck equation. Assuming the initial distributions in (4.2) are probability measures and $d \geq 2$, Kotelenez and Kurtz (2006) show that the stochastic perturbation is "small" (in a certain topology) provided the correlation length between the correlated Brownian motions is "small". Consequently, for small correlation length the deterministic Fokker-Planck equation is a good approximation to the stochastic Fokker-Planck equation.■

Without loss of generality, we assume $s = 0$ in what follows and we suppress the dependence of the coefficients F and $\mathcal{J}$ on t in the formulas. Suppose

$$\mathcal{X}_0 \in L_{0,\mathcal{F}_0}(\mathbf{M}_{\varpi,\infty}). \tag{4.3}$$

Let $b > 0$ be a constant and set for a measure $\mu \in \mathbf{M}_{\infty,\varpi}$

$$\mu_b := \begin{cases} \mu, & \text{if} \quad \gamma_\varpi(\mu) < b\ , \\ \frac{b}{\gamma_\varpi(\mu)}\mu, & \text{if} \quad \gamma_\varpi(\mu) \geq b. \end{cases} \tag{4.4}$$

This truncation allows us to define recursively solutions of (4.1) as follows: Let $r_1(t, \mathcal{X}_{0,b}, \omega, q)$ be the unique solution of (2.7), driven by the input process $\mathcal{X}_{0,b}$. Henceforth, we will tacitly assume that the solutions of (2.7) are the space-time measurable versions, whose existence follows from Part 3) of Theorem 2.1. For simplicity of notation, we drop the "bar" over those versions. Define the empirical measure process associated with $r_1(t, ...)$ as follows:

$$\mathcal{X}_1(t, b, \omega) := \int \delta_{r_1(t,\mathcal{X}_{0,b},\omega,q)} \mathcal{X}_{0,b}(\omega, dq).$$

$\mathcal{X}_1(t, b, \omega)$ is continuous and $\mathcal{F}_t$-adapted, whence the same holds for $\gamma_\varpi(\mathcal{X}_1(t, b, \omega))$. Moreover, by the boundedness of the coefficients of (2.7), $\varphi(t, q) := r_1(t, \mathcal{X}_{0,b}, \omega, q)$ satisfies the assumptions of Proposition 3.2. Therefore, (3.2) implies

$$E \sup_{0\leq t\leq T} \gamma_\varpi^2(\mathcal{X}_1(t, b)) \leq c_T E\gamma_\varpi^2(\mathcal{X}_{0,b}) \leq c_T b.$$

Hence, $\mathcal{X}_1(t, b) \in L_{2,\mathcal{F}}(C([0, T]; \mathbf{M}_{\infty,\varpi}))$ and we can define $r_2(t, \mathcal{X}_1(b), \omega, q)$ as the unique solution of (2.7), driven by the input process $\mathcal{X}_1(t, b)$. Suppose we have already defined $r_m(t, \mathcal{X}_{m-1}(b), \omega, q)$ as (unique) solutions of (2.7) with empirical measure processes $\mathcal{X}_{m-1}(t, b) \in L_{2,\mathcal{F}}(C([0, T]; \mathbf{M}_{\infty,\varpi}))$ for $m = 1, .., n$. As before, the solution $r_n(t, \mathcal{X}_{n-1}(b), \omega, q$ satisfies the assumptions of Proposition 2.3 and we define the empirical measure process of $r_n(t, \mathcal{X}_{n-1}(b), \omega, q)$ by

$$\mathcal{X}_n(t, b, \omega) := \int \delta_{r_n(t,\mathcal{X}_{n-1,b},\omega,q)} \mathcal{X}_{0,b}(\omega, dq). \tag{4.5}$$

Again (3.2) implies

$$E \sup_{0\leq t\leq T} \gamma_\varpi^2(\mathcal{X}_n(t, b)) \leq c_T E\gamma_\varpi^2(\mathcal{X}_{0,b}), \tag{4.6}$$

where the constant c_T depends only on T and the coefficients of (2.7).

Set

$$\Omega_b := \{\omega : \gamma_\varpi(\mathcal{X}_0(\omega)) \le b\}. \tag{4.7}$$

Clearly,

$$\Omega_{b_2} \supset \Omega_{b_1}, \quad \text{if } b_2 \ge b_1 > 0 \ .$$

Proposition 4.2

Suppose $b_2,\ b_1 > 0$. Then, a.s., for all n

$$\mathcal{X}_n(\cdot, b_2, \omega) 1_{\Omega_{b_1} \cap \Omega_{b_2}} \equiv \mathcal{X}_n(\cdot, b_1, \omega) 1_{\Omega_{b_1} \cap \Omega_{b_2}} \tag{4.8}$$

Proof

(i) The fact that $\Omega_b \in \mathcal{F}_0$ allows us to apply (2.10) with the conditional expectation with respect to the events from $\mathcal{F}_0$. Suppose $b_2 \ge b_1 > 0$. Hence,

$$\left.\begin{array}{c} E \sup_{0 t \le T} \rho^2(r_n(t, \mathcal{X}_{n-1}(b_2), q) - r_n(t, \mathcal{X}_{n-1}(b_1), q)) 1_{\Omega_{b_1}} \\ \le c_T \int_0^T \gamma_\varpi^2(\mathcal{X}_{n-1}(s, b_2) - \mathcal{X}_{n-1}(s, b_1)) 1_{\Omega_{b_1}} ds \\ = c_T \int_0^T \gamma_\varpi^2(\mathcal{X}_{n-1}(s, b_2) 1_{\Omega_{b_1}} - \mathcal{X}_{n-1}(s, b_1) 1_{\Omega_{b_1}}) 1_{\Omega_{b_1}} ds, \end{array}\right\} \tag{4.9}$$

where the last identity is obvious, as for $\omega \notin \Omega_{b_1}$ the expression $\gamma_\varpi^2(\mathcal{X}_{n-1}(s, b_2) - \mathcal{X}_{n-1}(s, b_1)) 1_{\Omega_{b_1}} = 0$.

(ii) We show by induction that the right hand side of (4.9) equals 0. First of all, note that

$$\mathcal{X}_{0,b_2} 1_{\Omega_{b_1}} = \mathcal{X}_{0,b_1} 1_{\Omega_{b_1}},$$

because both sides equal $\mathcal{X}_0 1_{\Omega_{b_1}}$. If

$$\mathcal{X}_{n-1}(\cdot, b_2, \omega) 1_{\Omega_{b_1}} \equiv \mathcal{X}_{n-1}(\cdot, b_1, \omega) 1_{\Omega_{b_1}}.$$

then (4.9) implies a.s.

$$r_n(\cdot, \mathcal{X}_{n-1}(b_2), q) 1_{\Omega_{b_1}} \equiv r_n(\cdot, \mathcal{X}_{n-1}(b_1), q) 1_{\Omega_{b_1}}.$$

Therefore,

$$\int \delta_{r_n(\cdot, \mathcal{X}_{n-1}(b_2), \omega, q)} \mathcal{X}_{0,b_2}(\omega, dq) 1_{\Omega_{b_1}}(\omega) \equiv \int \delta_{r_n(\cdot, \mathcal{X}_{n-1}(b_1), \omega, q)} \mathcal{X}_{0,b_1}(\omega, dq) 1_{\Omega_{b_1}}(\omega),$$

whence by (4.5)

$$\mathcal{X}_n(\cdot, b_2, \omega) 1_{\Omega_{b_1}} \equiv \mathcal{X}_n(\cdot, b_1, \omega) 1_{\Omega_{b_1}}.$$

∎

For a second truncation, let $c > 0$ and define stopping times as follows:

$$\tau_n(c,b,\omega) := \inf\{t \ge 0 : \gamma_\varpi(\mathcal{X}_n(t,b,\omega)) \ge c\} \quad \text{and} \quad \tau(c,b,\omega) := \inf_n \tau_n(c,b,\omega). \tag{4.10}$$

By (4.8) we have for $b_2, b_1 > 0$ and $\forall c > 0$

$$\tau_n(c,b_2,\omega)_{|\Omega_{b_1}\cap\Omega_{b_2}} = \tau_n(c,b_1,\omega)_{|\Omega_{b_1}\cap\Omega_{b_2}} \quad \text{and} \quad \tau(c,b_2,\omega)_{|\Omega_{b_1}\cap\Omega_{b_2}} = \tau(c,b_1,\omega)_{|\Omega_{b_1}\cap\Omega_{b_2}}. \tag{4.11}$$

Lemma 4.3

$$P\{\omega : \lim_{c\to\infty} \tau(c,b,\omega) = \infty\} = 1 \quad \forall b > 0. \tag{4.12}$$

Proof

Suppose (4.12) is not correct and note that $\tau(c,b)$ is monotone increasing as c increases. Therefore, there must be a $T > 0$, a $b > 0$ and a $\delta > 0$ such that

$$P(\cap_{m\in\mathbf{N}}\{\omega : \tau(m,b,\omega) < T\}) \ge \delta.$$

Set

$$\underline{\tau}_n(c,b) := \tau_1(c,b) \wedge \tau_2(c,b) \wedge .... \wedge \tau_n(c,b).$$

Then

$$\underline{\tau}_n(c,b) \downarrow \tau(c,b) \ \text{ a.s., as } n \longrightarrow \infty \ .$$

Thus,

$$\cap_{m\in\mathbf{N}}\{\omega : \tau(m,b,\omega) < T\} = \cap_{n\in\mathbf{N}} \cap_{m\in\mathbf{N}} \{\omega : \underline{\tau}_n(m,b,\omega) < T\}.$$

As

$$B_n := \cap_{m\in\mathbf{N}}\{\omega : \underline{\tau}_n(m,b,\omega) < T\} \uparrow, \ \text{ as } n \longrightarrow \infty \ ,$$

there must be an $\bar{n}$ such that

$$P(B_{\bar{n}}) \ge \frac{\delta}{2}.$$

The definition of the stopping times implies

$$B_n \subset \cap_{m\in\mathbf{N}}\{\omega : \max_{k=1,\dots,n} \sup_{0\le t\le T} \gamma_\varpi(X_k(t,b,\omega)) \ge m\}.$$

Set

$$A_m(n) := \{\omega : \max_{k=1,\dots,n} \sup_{0\le t\le T} \gamma_\varpi(X_k(t,b,\omega)) \ge m\}.$$

Obviously, the $A_m(n)$ are monotone decreasing as m increases. Therefore, the assumption on $P(B_{\bar{n}})$ implies

$$P(\cap_{m\in\mathbf{N}}A_m(\bar n)) \ge \frac{\delta}{2}.$$

However, by (4.6), a.s. $\forall k$,

$$\sup_{0\le t\le T}\gamma_\varpi(X_k(t,b,\omega)) < \infty.$$

Hence, we have also a.s. $\forall n$

$$\max_{k=1,\ldots,n}\sup_{0\le t\le T}\gamma_\varpi(X_k(t,b,\omega)) < \infty.$$

This implies

$$\forall n \quad P(A_m(n)) \downarrow 0, \quad \text{as } m \longrightarrow \infty ,$$

contradicting $P(\cap_{m\in\mathbf{N}}A_m(\bar n)) \ge \frac{\delta}{2}$. ■

Abbreviate in what follows

$$\left.\tau := \tau(c,b), \quad r_i(t,q) := r_i(t,\mathcal{X}_{i-1},b,q).\ \right\} \tag{4.13}$$

Lemma 4.4

For any $T>0$ there is a finite constant $c_T := c(T,F,\mathcal{J},c,b,\gamma)$ such that $\forall t\in[0,T]$

$$\left.\begin{array}{c} E\int\sup_{0\le s\le t}\rho^4(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))[\varpi(r_n(s\wedge\tau,q))+\varpi(r_m(s\wedge\tau,q))]\mathcal{X}_{0,b}(dq)\\ \le c_T\int_0^t E\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))ds \end{array}\right\} \tag{4.14}$$

Proof

(i) Let $\chi\in C_b^2(\mathbf{R};\mathbf{R})$ be an odd function such that there are constants $0<c_1\le c_2<\infty$ and the following properties hold:

$$\left.\begin{array}{c} \chi(u)=u \ \ \forall u\in[-1,1], \quad \chi'(u)>0 \ \ \forall u\in\mathbf{R},\\ |\chi''(u)u^2|\le c_1|\chi'(u)u|\le c_2|\chi(u)| \ \ \forall u\in\mathbf{R},\\ |||\chi|||\le 2, \quad |||\chi'|||\le 1, \end{array}\right\} \tag{4.15}$$

where χ' and χ'' are the first and second derivatives of χ, respectively. Then

$$\rho^2(r)\le\chi(|r|^2)\le 2\rho^2(r). \tag{4.16}$$

(ii) Abbreviate

$$f(s,q) := r_n(t\wedge\tau,q)-r_m(t\wedge\tau,q), \quad g(t,q) := r_i(t,q), \quad i\in\{n,m\}. \tag{4.17}$$

Note that $f(0,q)=0$ and $g(0,q)=q$, whence we have $\chi(|f(0,q)|^2)=0$ and $\varpi(g(0,q))=\varpi(q)$. Itô's formula yields

$$\left.\begin{array}{l}\chi(|f(t,q)|^2)=\int_0^t\chi'(|f(s,q)|^2)2f(s,q)\bullet df(s,q)\\ \qquad+\int_0^t(\chi''(|f(s,q)|^2)2\sum_{k,\ell=1}^d f_k(s,q)f_\ell(s,q)+\chi'(|f(s,q)|^2)\delta_{k\ell})d[f_k(s,q),f_\ell(s,q)],\end{array}\right\}\quad(4.18)$$

and for $\varpi_\beta(q):=\varpi^\beta(q)$, $\beta>0$,

$$\left.\begin{array}{l}\varpi_\beta(g(t,q))=\varpi_\beta(q)+\int_0^t(\nabla\varpi_\beta)(g(s,q))\bullet dg(s,q)\\ \qquad+\frac{1}{2}\sum_{k,\ell=1}^d\int_0^t(\partial^2_{k\ell}\varpi_\beta)(g(s,q))d[g_k(s,q),g_\ell(s,q)].\end{array}\right\}\quad(4.19)$$

Next, we apply Itô's formula to the product of the left hand sides of (4.18) and (4.19):

$$\left.\begin{array}{l}\chi(|f(t,q)|^2)\varpi_\beta(g(t,q))\\ =\int_0^t\varpi_\beta(g(s,q))d\chi(|f(s,q)|^2)+\int_0^t\chi(|f(s,q)|^2)d\varpi_\beta(g(s,q))+\int_0^t d[\chi(|f(s,q)|^2),\varpi_\beta(g(s,q))].\end{array}\right\}\quad(4.20)$$

In what follows, we first derive bounds for the deterministic differentials in (4.20) employing (2.9). By (4.18)

$$\left.\begin{array}{l}d\chi(|f(s,q)|^2)=\chi'(|f(s,q)|^2)2f(s,q)\bullet df(s,q)\\ \qquad+(\chi''(|f(s,q)|^2)2\sum_{k,\ell=1}^d f_k(s,q)f_\ell(s,q)+\chi'(|f(s,q)|^2)\delta_{k\ell})d[f_k(s,q),f_\ell(s,q)].\end{array}\right\}\quad(4.21)$$

First,

$$\begin{array}{l}df(s,q)=(F(r_n(s\wedge\tau,q),\mathcal{X}_{n-1}(b))-F(r_m(s\wedge\tau,q),\mathcal{X}_{m-1}(b)))ds\\ \qquad+\int(\mathcal{J}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(b))-\mathcal{J}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(b)))w(dp,ds).\end{array}$$

Recall that by (4.10) $\varpi_\beta(\mathcal{X}_{i-1}(t\wedge\tau,b))\le c\ \ \forall i\in\mathbf{N}$. Let $\tilde c$ be a positive constant, depending on $c,F,\mathcal{J},T,b,\beta$ and γ_ϖ, and the value of $\tilde c$ may be different for different steps in the following estimates. Therefore, (4.11) yields

$$\left.\begin{array}{l}|F(r_n(s\wedge\tau,q),\mathcal{X}_{n-1}(b))-F(r_m(s\wedge\tau,q),\mathcal{X}_{m-1}(b))|\\ \le\tilde c(\rho(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))+\gamma_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))).\end{array}\right\}\quad(4.22)$$

Again by (2.9),

$$\left.\begin{array}{c}d[f_k(s,q),f_\ell(s,q)]\\ =\sum_{j=1}^d\int(\mathcal{J}_{kj}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(s\wedge\tau,b))-\mathcal{J}_{kj}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(s\wedge\tau,b)))\times\\ \times(\mathcal{J}_{\ell j}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(b))-\mathcal{J}_{\ell j}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(b)))dpds\\ \le\tilde c(\rho^2(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))+\gamma^2_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b)))ds\\ \\ \text{and also}\\ \\ \sum_{j=1}^d\int(\mathcal{J}_{kj}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(s\wedge\tau,b))-\mathcal{J}_{kj}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(s\wedge\tau,b)))\times\\ \times(\mathcal{J}_{\ell j}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(b))-\mathcal{J}_{\ell j}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(b)))dpds\\ \le\tilde c(\rho(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))+\gamma_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b)))ds,\end{array}\right\}\quad(4.23)$$

where in the last inequality we used the boundedness of the coefficients. Employing (4.22) and (4.23), (4.15) yields the following estimates:

$$\left.\begin{array}{c}|\chi'(|f(s,q)|^2)2f(s,q)|\rho(f(s,q)) \le \tilde{c}\chi(|f(s,q)|^2),\\ |\chi'(|f(s,q)|^2)2f(s,q)| \le \tilde{c}\\ |(\chi''(|f(s,q)|^2)2\sum_{k,\ell=1}^d f_k(s,q)f_\ell(s,q)| \le \tilde{c}\chi(|f(s,q)|^2),\\ |\chi'(|f(s,q)|^2)| \le 1,\\ |\chi''(|f(s,q)|^2)2\sum_{k,\ell=1}^d f_k(s,q)f_\ell(s,q)|d[f_k(s,q),f_\ell(s,q)]\\ \le \tilde{c}\chi(|f(s,q)|^2)[\rho^2(f(s,q)) + \gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))]ds,\\ \chi'(|f(s,q)|^2))\sum_{k=1}^d d[f_k(s,q),f_k(s,q)] \le \tilde{c}\{\chi(|f(s,q)|^2)ds + \gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))ds\},\end{array}\right\} \tag{4.24}$$

where the last inequality follows from

$$\begin{array}{c}\chi'(|f(s,q)|^2))\sum_{k=1}^d d[f_k(s,q),f_k(s,q)]\\ \le \chi'(|f(s,q)|^2)\tilde{c}(\rho^2(r_n(s\wedge\tau,q) - r_m(s\wedge\tau,q)) + \gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b)))ds\\ \le \tilde{c}\chi(|f(s,q)|^2)ds + \tilde{c}\gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))ds.\end{array}$$

We note that

$$|\chi(|f(s,q)|^2) \le 2.$$

Therefore, by (4.21), (4.22) and (4.24),

$$\left.\begin{array}{c}d\chi(|f(s,q)|^2)\\ \le \tilde{c}(\chi(|f(s,q)|^2) + \gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b)))ds\\ +\chi'(|f(s,q)|^2)2f(s,q)\bullet\int(\mathcal{J}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(b)) - \mathcal{J}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(b)))w(dp,ds).\end{array}\right\} \tag{4.25}$$

Set

$$dm_1(s,q) := \int(\mathcal{J}(r_n(s\wedge\tau,q),p,\mathcal{X}_{n-1}(b)) - \mathcal{J}(r_m(s\wedge\tau,q),p,\mathcal{X}_{m-1}(b)))w(dp,ds). \tag{4.26}$$

Hence,

$$\left.\begin{array}{l}\int_0^t \varpi_\beta(g(s,q))d\chi(|f(s,q)|^2)\\ \le \tilde{c}\int_0^t \varpi_\beta(g(s,q))\chi(|f(s,q)|^2) + \varpi_\beta(g(s,q))\gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b)))ds\\ + \int_0^t \varpi_\beta(g(s,q))\chi'(|f(s,q)|^2)2f(s,q)\bullet dm_1(s,q).\end{array}\right\} \tag{4.27}$$

Next, by (4.19)

$$\left.\begin{array}{l} d\varpi_\beta(g(s,q)) = (\nabla\varpi_\beta)(g(s,q)) \bullet dg(s,q) + \frac{1}{2}\sum_{k,\ell=1}^d (\partial^2_{k\ell}\varpi_\beta)(g(s,q))d[g_k(s,q), g_\ell(s,q)] \\ \qquad\qquad\quad = (\nabla\varpi_\beta)(r_i(s\wedge\tau,q)) \bullet F(r_i(s\wedge\tau,q), \mathcal{X}_{i-1}(b))ds \\ \qquad\qquad\quad +\frac{1}{2}\sum_{k,\ell=1}^d (\partial^2_{k\ell}\varpi_\beta)(r_i(s\wedge\tau,q))d[r_{i,k}(s\wedge\tau,q), r_{i,\ell}(s\wedge\tau,q)] \\ \qquad\qquad\quad +(\nabla\varpi_\beta)(r_i(s\wedge\tau,q)) \bullet \int \mathcal{J}(r_i(s\wedge\tau,q)-p, \mathcal{X}_{i-1}(b))w(dp,ds). \end{array}\right\} \quad (4.28)$$

By (2.9), (5.1) and the definition of τ

$$\left.\begin{array}{l} |\nabla\varpi_\beta)(r_i(s\wedge\tau,q))||F(r_i(s\wedge\tau,q), \mathcal{X}_{i-1}(b))| \le \tilde{c}\varpi_\beta(r_i(s\wedge\tau,q)), \\ \frac{1}{2}\sum_{k,\ell=1}^d (\partial^2_{k\ell}\varpi_\beta(r_i(s\wedge\tau,q))d[r_{i,k}(s\wedge\tau,q), r_{i,\ell}(s\wedge\tau,q)] \le \tilde{c}\varpi_\beta(r_i(s\wedge\tau,q)). \end{array}\right\} \quad (4.29)$$

Abbreviating

$$dm_2(s,q) := \int \mathcal{J}(g(s,q), p, \mathcal{X}_{i-1}(b))w(dp,ds), \quad (4.30)$$

we obtain

$$\left.\begin{array}{l} |\int_0^t \chi(|f(s,q)|^2)d\varpi_\beta(g(s,q))| \\ \le \tilde{c}\int_0^t \chi(|f(s,q)|^2)\varpi_\beta(g(s,q))ds + \int_0^t \chi(|f(s,q)|^2)(\nabla\varpi_\beta)(g(s,q)) \bullet dm_2(s,q). \end{array}\right\} \quad (4.31)$$

Finally,

$$[\chi(|f(t,q)|^2), \varpi_\beta(g(t,q))] = [\int_0^t \chi'(|f(s,q)|^2)2f(s,q) \bullet dm_1(s), \int_0^t (\nabla\varpi_\beta)(g(s,q)) \bullet dm_2(s)]. \quad (4.32)$$

Hence,

$$\left.\begin{array}{l} [\chi(|f(t,q)|^2), \varpi_\beta(g(t,q))] \\ = \sum_{k,\ell=1}\int_0^t \chi'(|f(s,q)|^2)2f_k(s,q)(\partial_\ell\varpi_\beta)(g(s,q))d[m_{1,k}(s), m_{2,\ell}(s)]. \end{array}\right\} \quad (4.33)$$

We apply (4.23) and the Cauchy-Schwarz inequality in addition to (2.9) and obtain

$$d[m_{1,k}(s), m_{2,\ell}(s)] \le \tilde{c}(\rho(f(s,q)) + \gamma_\varpi(\mathcal{X}_{n-1}(b) - \mathcal{X}_{m-1}(b)))ds. \quad (4.34)$$

Consequently, by (5.1) and (4.16) as well as by $|f_j(s,q)| \le |f(s,q)|$,

$$\left.\begin{array}{c} |[\chi(|f(t,q)|^2), \varpi_\beta(g(t,q))]| \\ \le \tilde{c}\int_0^t |\chi'(|f(s,q)|^2)||f(s,q)|\varpi_\beta(g(s,q))(\rho(f(s,q)) + \gamma_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b)))ds \\ \le \tilde{c}\int_0^t \chi(|f(s,q)|^2)\varpi_\beta(g(s,q))ds \\ +\tilde{c}\int_0^t \varpi_\beta(g(s,q))|\chi'(|f(s,q)|^2)||f(s,q)|\gamma_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))ds \\ \le \tilde{c}\int_0^t \chi(|f(s,q)|^2)\varpi_\beta(g(s,q))ds \\ +\tilde{c}\int_0^t \varpi_\beta(g(s,q))[|\chi'(|f(s,q)|^2)|^2|f(s,q)|^2 + \gamma^2_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))]ds \\ \text{(again employing } ab \le \frac{1}{2}(a^2+b^2)\text{)} \\ \le \tilde{c}\int_0^t \chi(|f(s,q)|^2)\varpi_\beta(g(s,q))ds \\ +\tilde{c}\int_0^t \varpi_\beta(g(s,q))[\chi(|f(s,q)|^2) + \gamma^2_\varpi(\mathcal{X}_{n-1}(s\wedge\tau,b) - \mathcal{X}_{m-1}(s\wedge\tau,b))]ds \\ \text{(because } |\chi'(|f(s,q)|^2)|^2|f(s,q)|^2 \le |\chi'(|f(s,q)|^2)|\chi(|f(s,q)|^2) \le \tilde{c}\chi(|f(s,q)|^2)\text{ .)} \end{array}\right\} \quad (4.35)$$

From the previous calculations we obtain

$$\left.\begin{array}{c}\chi(|f(t,q)|^2)\varpi_\beta(g(t,q))\\ \leq \tilde{c}\int_0^t[\chi(|f(s,q)|^2)\varpi_\beta(g(s,q))+\varpi_\beta(g(s,q))\gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))]ds\\ +\int_0^t\varpi_\beta(g(s,q))\chi'(|f(s,q)|^2)2f(s,q)dm_1(s,q)+\int_0^t\chi(|f(s,q)|^2)(\nabla\varpi_\beta)(g(s,q))\bullet dm_2(s,q).\end{array}\right\} \quad (4.36)$$

We next derive bounds for the quadratic variations of the martingales. Doob's inequality yields

$$E\sup_{0\leq s\leq t}\{\int_0^s\varpi_\beta(g(u,q))\chi'(|f(s,q)|^2)2f(s,q)\bullet dm_1(u,q)\}^2\leq E\int_0^t\varpi_\beta^2(g(u,q))(\chi'(|f(s,q)|^2)2|f(s,q)|)^2d[m_1(u,q)]$$

By (4.24), (4.23) and (4.16),

$$\left.\begin{array}{c}d[\chi'(|f(s,q)|^2)2f(s,q)\bullet m_1(s,q)]\leq 4|\chi'(|f(s,q)|^2)||f(s,q)|^2\sum_{k,\ell=1}^d d[f_k(s,q),f_\ell(s,q)]\\ \leq\tilde{c}\chi(|f(s,q)|^2)(\rho^2(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))+\gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b)))ds\\ \leq\tilde{c}\{\chi^2(|f(s,q)|^2)+\chi(|f(s,q)|)\gamma_\varpi^2(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))\}ds\\ \leq\tilde{c}\{\chi^2(|f(s,q)|^2)+\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))\}ds\\ \text{(employing } ab\leq\frac{1}{2}(a^2+b^2)\text{ in the last inequality).}\end{array}\right\} \quad (4.37)$$

Thus,

$$\left.\begin{array}{c}E\sup_{0\leq s\leq t}|\int_0^s\varpi_\beta(g(u,q))\chi'(|f(s,q)|^2)2f(s,q)\bullet dm_1(u,q)|^2\\ \leq\tilde{c}E\int_0^t\varpi_\beta^2(g(u,q))\{\chi^2(|f(u,q)|^2)+\gamma_\varpi^4(\mathcal{X}_{n-1}(u\wedge\tau,b)-\mathcal{X}_{m-1}(u\wedge\tau,b))\}du.\end{array}\right\} \quad (4.38)$$

It is now convenient to square the terms in (4.20). (4.27) and (4.38) imply

$$\left.\begin{array}{c}E\sup_{0\leq s\leq t}|\int_0^s\varpi_\beta(g(u,q))d\chi(|f(u,q)|^2)|^2\\ \leq\tilde{c}E\int_0^t\varpi_\beta^2(g(u,q))\{\chi^2(|f(u,q)|^2)+\gamma_\varpi^4(\mathcal{X}_{n-1}(u\wedge\tau,b)-\mathcal{X}_{m-1}(u\wedge\tau,b))\}du.\end{array}\right\} \quad (4.39)$$

Similarly, employing (2.9) (i.e., boundedness of $\mathcal{J}$) and (5.1), Doob's inequality implies

$$\left.\begin{array}{c}E\sup_{0\leq s\leq t}|\int_0^s\chi(|f(u,q)|^2)(\nabla\varpi_\beta)(g(u,q))\bullet dm_2(u,q)|^2\\ \leq\tilde{c}E\int_0^t\chi^2(|f(u,q)|^2)\varpi_\beta^2(g(u,q))du.\end{array}\right\} \quad (4.40)$$

Hence, by (4.31) and (4.40)

$$\left.\begin{array}{c}E\sup_{0\leq s\leq t}(\int_0^s\chi(|f(u,q)|^2)d\varpi_\beta(g(u,q)))^2\\ \leq\tilde{c}E\int_0^t\chi^2(|f(u,q)|^2)\varpi_\beta^2(g(u,q))du.\end{array}\right\} \quad (4.41)$$

We also square both sides in (4.35) and employ the Cauchy-Schwarz inequality:

$$\left.\begin{array}{l}[\chi(|f(t,q)|^2),\varpi_\beta(g(t,q))]^2\\ \leq\tilde{c}\int_0^t\chi^2(|f(s,q)|^2)\varpi_\beta^2(g(s,q))ds+\tilde{c}\int_0^t\varpi_\beta^2(g(s,q))\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))ds.\end{array}\right\} \quad (4.42)$$

Choosing $\beta := \frac{1}{2}$, the previous steps imply

$$\left.\begin{array}{l} E\sup_{0\le s\le t}\chi^2(|f(t,q)|^2)\varpi(g(t,q)) \\ \le \tilde{c}E\int_0^t[\varpi(g(u,q))\chi^2(|f(u,q)|^2)+\varpi(g(u,q))\gamma_\varpi^4(\mathcal{X}_{n-1}(u\wedge\tau,b)-\mathcal{X}_{m-1}(u\wedge\tau,b))]du. \end{array}\right\} \quad (4.43)$$

By estimate (4.16) this implies

$$\left.\begin{array}{l} E\sup_{0\le s\le t}\rho^4(f(t,q))\varpi(g(t,q)) \\ \le \tilde{c}E\int_0^t[\sup_{0\le u\le s}\varpi(g(u,q))\rho^4(f(u,q)))+\varpi(g(s,q))\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))]ds. \end{array}\right\} \quad (4.44)$$

Gronwall's inequality implies

$$E\sup_{0\le s\le t}\rho^4(f(t,q))\varpi(g(t,q)) \le c_T\int_0^t E\varpi(g(s,q))\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))ds. \quad (4.45)$$

Adding up the inequalities of (4.45) for $g(t,q)=r_n(t\wedge\tau,q)$ and for $g(t,q)=r_m(t\wedge\tau,q)$ and integrating against $\mathcal{X}_{0,b}(dq)$ yields

$$\left.\begin{array}{l} E\int\sup_{0\le s\le t}\rho^4(r_n(s\wedge\tau,q)-r_m(s\wedge\tau,q))[\varpi(r_n(s\wedge\tau,q))+\varpi(r_m(s\wedge\tau,q))]\mathcal{X}_{0,b}(dq) \\ \le c_T\int_0^t E\int[\varpi(r_n(s\wedge\tau,q))+\varpi(r_m(s\wedge\tau,q))]\mathcal{X}_{0,b}(dq)\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))ds \end{array}\right\} \quad (4.46)$$

Taking into account that

$$\int[\varpi(r_n(s\wedge\tau,q))+\varpi(r_m(s\wedge\tau,q))]\mathcal{X}_{0,b}(dq)=\gamma_\varpi(\mathcal{X}_n(s\wedge\tau,b))+\gamma_\varpi(\mathcal{X}_m(s\wedge\tau,b))\le 2c$$

we obtain (4.14) . ∎

Proposition 4.5

For any $T>0$ there is a finite constant $c_T := c(T,F,\mathcal{J},c,b,\gamma)$ such that

$$E\sup_{0\le t\le T}\gamma_\varpi^4(\mathcal{X}_n((t\wedge\tau,b)-\mathcal{X}_m(t\wedge\tau,b)))\le c_T\int_0^T E\sup_{0\le s\le t}\gamma_\varpi^4(\mathcal{X}_{n-1}(s\wedge\tau,b)-\mathcal{X}_{m-1}(s\wedge\tau,b))dt. \quad (4.47)$$

Proof

$$\begin{array}{l} E\sup_{0\le t\le T}\gamma_\varpi^4(\mathcal{X}_n((t\wedge\tau,b)-\mathcal{X}_m(t\wedge\tau,b)) \\ = E\sup_{0\le t\le T}\sup_{\|f\|_{L,\infty}\le 1}\times \\ \times\{\int[f(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q))\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q))-f(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q))\varpi(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q))]\mathcal{X}_{0,b}(dq)\}^4. \end{array}$$

Further,

$$\left.\begin{array}{c} |f(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q))\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) - f(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))\varpi(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))| \\ \le |f(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) - f(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))|\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) \\ +|f(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))||\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) - \varpi(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))| \\ \le \rho(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) \\ +\gamma\rho(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))[\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))] \\ \text{(by the definition of } f \text{ and (5.1))} \\ \le (\gamma+1)\rho(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))[\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))]. \end{array}\right\} \tag{4.48}$$

Thus,

$$\left.\begin{array}{c} E\sup_{0\le t\le T}\gamma_\varpi^4\mathcal{X}_n((t\wedge\tau,b) - \mathcal{X}_m(t\wedge\tau,b)) \\ \le (\gamma+1)^4 E\sup_{0\le t\le T}\{\int\rho(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))\times \\ \times(\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q)))\mathcal{X}_{0,b}(dq)\}^4 \\ \le (\gamma+1)^4 E\sup_{0\le t\le T}\int\rho^4(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))\times \\ \times(\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q)))\mathcal{X}_{0,b}(dq)\times \\ \times\{\int(\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q)))\mathcal{X}_{0,b}(dq)\}^3 \\ \text{(by Hölder's inequality)} \\ \le \tilde{c}\int E\sup_{0\le t\le T}\rho^4(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q) - r_m(t\wedge\tau,\mathcal{X}_{m-1}(b),q))\times \\ \times(\varpi(r_n(t\wedge\tau,\mathcal{X}_{n-1}(b),q)) + \varpi(r_m(t\wedge\tau,\mathcal{X}_{n-1}(b),q)))\mathcal{X}_{0,b}(dq) \\ \text{(because } \int\varpi(r_i(t\wedge\tau,\mathcal{X}_{i-1}(b),q))\mathcal{X}_{0,b}(dq) = \gamma_\varpi(\mathcal{X}_i(t\wedge\tau,b)) \le c, \quad i=m,n,) \;. \end{array}\right\} \tag{4.49}$$

The proof follows from (4.14). ∎

Corollary 4.6

There is a unique $\bar{\mathcal{X}}(\cdot) \in L_{loc,4,\mathcal{F}}(C((0,T];\mathbf{M}_{\infty,\varpi}))$, and for any positive rational numbers b and c and stopping time $\tau = \tau(c,b)$

$$E\sup_{0\le t\le T}\gamma_\varpi^4(\mathcal{X}_n(t\wedge\tau(c,b),b)1_{\Omega_b} - \bar{\mathcal{X}}(t\wedge\tau(c,b))1_{\Omega_b}) \longrightarrow 0, \quad \text{as } n\longrightarrow\infty \;. \tag{4.50}$$

Proof

The space $L_{loc,4,\mathcal{F}}(C((0,T];\mathbf{M}_{\infty,\varpi}))$ is complete. Therefore, we obtain from Proposition 4.4 a unique $\tilde{\mathcal{X}}(\cdot,c,b)$ such that

$$E\sup_{0\le t\le T}\gamma_\varpi^4(\mathcal{X}_n(t\wedge\tau(c,b),b) - \tilde{\mathcal{X}}(t,c,b)) \longrightarrow 0, \quad \text{as } n\longrightarrow\infty \;. \tag{4.51}$$

Further, the monotonicity of $\tau(c,b)$ in c implies that for $c_2 \ge c_1$ and $b > 0$

$$\mathcal{X}_n(\cdot\wedge\tau(c_2,b)\wedge\tau(c_1,b),b,\omega) \equiv \mathcal{X}_n(\cdot\wedge\tau(c_1,b),b,\omega) \quad \text{a.e. .}$$

Hence,

$$\tilde{\mathcal{X}}(\cdot,c_2,b,\omega)1_{\{t\le\tau(c_1,b)\}} \equiv \tilde{\mathcal{X}}(\cdot,c_1,b,\omega)1_{\{t\le\tau(c_1,b)\}} \quad \text{a.e. .} \tag{4.52}$$

Let $\tau(0,b) \equiv 0$ and define

$$\bar{\mathcal{X}}(t,b,\omega) := \sum_{i\in\mathbf{N}} \tilde{\mathcal{X}}(t,i,b,\omega)1_{\{\tau(i-1,b)<t\le\tau(i,b)\}} \tag{4.53}$$

(4.52) implies that

$$\tilde{\mathcal{X}}(t\wedge\tau(c,b),i,b,\omega)1_{\{\tau(i-1,b)<t\le\tau(i,b)\}} \equiv \tilde{\mathcal{X}}(t\wedge\tau(c,b,c),b,\omega)1_{\{\tau(i-1,b)<t\le\tau(i,b)\}} \quad \text{a.e. , uniformly in } t\ . \tag{4.54}$$

Indeed, for $i \le c$, (4.54) follows since we consider only values of $t \in (\tau(i-1,b),\tau(i,b)]$. Therefore, we may apply (4.52) for $c = c_2$ and $i = c_1$. If, however, $i > c$, then the stopping at $\tau(c,b)$ restricts the values of t to those where $t \le \tau(c,b)$. Hence, (4.52) applies with $c_2 = i$ and $c_1 = c$. Thus,

$$\bar{\mathcal{X}}(t\wedge\tau(c,b),b) = \sum_{i\in\mathbf{N}} \tilde{\mathcal{X}}(t\wedge\tau(c,b),c,b)1_{\{\tau(i-1,b)<t\le\tau(i,b)\}} \quad \text{a.e. , uniformly in } t\ ,$$

whence,

$$\bar{\mathcal{X}}(t\wedge\tau(c,b),b) \equiv \tilde{\mathcal{X}}(t\wedge\tau(c,b),c,b) \equiv \tilde{\mathcal{X}}(t,c,b) \quad \text{a.e. , uniformly in } t\ . \tag{4.55}$$

Further, note that for $b_2 \ge b_1 > 0$ a.e. , uniformly in t ,

$$\begin{aligned}
&\tilde{\mathcal{X}}(t,i,b_2)1_{\{\tau(i-1,b_2)<t\le\tau(i,b_2)\}}1_{\Omega_{b_1}}\\
&= \tilde{\mathcal{X}}(t,i,b_2)1_{\{\tau(i-1,b_1)<t\le\tau(i,b_1)\}}1_{\Omega_{b_1}} \quad \text{(by (4.11))}\\
&= \lim_{n\to\infty}\mathcal{X}_n(t\wedge\tau(i,b_2),b_2)1_{\{\tau(i-1,b_1)<t\le\tau(i,b_1)\}}1_{\Omega_{b_1}} \quad \text{(by (4.51))}\\
&= \lim_{n\to\infty}\mathcal{X}_n(t\wedge\tau(i,b_1),b_1)1_{\{\tau(i-1,b_1)<t\le\tau(i,b_1)\}}1_{\Omega_{b_1}} \quad \text{(by (4.8) and (4.11))}\\
&= \tilde{\mathcal{X}}(t,i,b_1)1_{\{\tau(i-1,b_1)<t\le\tau(i,b_1)\}}1_{\Omega_{b_1}} \quad \text{(by (4.51))}
\end{aligned}$$

i.e., a.e. , uniformly in t,

$$\left.\begin{aligned}
\tilde{\mathcal{X}}(t,i,b_2)1_{\{\tau(i-1,b_2)<t\le\tau(i,b_2)\}}1_{\Omega_{b_1}} &= \tilde{\mathcal{X}}(t,i,b_1)1_{\{\tau(i-1,b_1)<t\le\tau(i,b_1)\}}1_{\Omega_{b_1}}\,,\\
\bar{\mathcal{X}}(t,b_2)1_{\Omega_{b_1}} &= \bar{\mathcal{X}}(t,b_1)1_{\Omega_{b_1}}\,,
\end{aligned}\right\} \tag{4.56}$$

where the second line follows from the first line and (4.53). We now set

$$\mathcal{X}(t) := \sum_{j\in\mathbf{N}} \bar{\mathcal{X}}(t,j)1_{\Omega_j\setminus\Omega_{j-1}}, \tag{4.57}$$

and we note that $\mathcal{X}(\cdot) \in L_{0,\mathcal{F}}(C([0,\infty);\mathbf{M}_{\infty,\varpi})) \cap_{p\ge1} L_{loc,p,\mathcal{F}}(C((0,\infty);\mathbf{M}_{\infty,\varpi}))$. We verify that a.e. uniformly in t,

$$\begin{aligned}
&\mathcal{X}(t\wedge\tau(c,b))1_{\Omega_b}\\
&= \textstyle\sum_{j\in\mathbf{N}}\bar{\mathcal{X}}(t\wedge\tau(c,b),j)1_{\Omega_j\setminus\Omega_{j-1}}1_{\Omega_b}\\
&= \textstyle\sum_{j\in\mathbf{N}}\sum_{i\in\mathbf{N}}\tilde{\mathcal{X}}(t\wedge\tau(c,b),i,j)1_{\{\tau(i-1,j)<t\le\tau(i,j)\}}1_{\Omega_j\setminus\Omega_{j-1}}1_{\Omega_b}\\
&= \textstyle\sum_{j\in\mathbf{N}}\sum_{i\in\mathbf{N}}\tilde{\mathcal{X}}(t\wedge\tau(c,b),c,j)1_{\{\tau(i-1,j)<t\le\tau(i,j)\}}1_{\Omega_j\setminus\Omega_{j-1}}1_{\Omega_b} \quad \text{(by (4.54))}\\
&= \textstyle\sum_{j\in\mathbf{N}}\sum_{i\in\mathbf{N}}\tilde{\mathcal{X}}(t\wedge\tau(c,b),c,b)1_{\{\tau(i-1,j)<t\le\tau(i,j)\}}1_{\Omega_j\setminus\Omega_{j-1}}1_{\Omega_b} \quad \text{(by (4.54))}\\
&= \textstyle\sum_{j\in\mathbf{N}}\sum_{i\in\mathbf{N}}\tilde{\mathcal{X}}(t\wedge\tau(c,b),i,b)1_{\{\tau(i-1,j)<t\le\tau(i,j)\}}1_{\Omega_j\setminus\Omega_{j-1}}1_{\Omega_b} \quad \text{(by (4.54))}\\
&= \bar{\mathcal{X}}(t\wedge\tau(c,b),b)1_{\Omega_b} \quad \text{(by (4.53))}\\
&= \tilde{\mathcal{X}}(t\wedge\tau(c,b),c,b)1_{\Omega_b} \quad \text{(by (4.55))}\ ,
\end{aligned}$$

i.e., for any positive rational numbers b and c

$$\mathcal{X}(t\wedge\tau(c,b))1_{\Omega_b}=\tilde{\mathcal{X}}(t\wedge\tau(c,b),b)1_{\Omega_b}=\tilde{\mathcal{X}}(t\wedge\tau(c,b),c,b)1_{\Omega_b}\quad \text{a.e. , uniformly in } t\,. \tag{4.58}$$

(4.58) in addition to (4.51) implies (4.50). ■

Let $r(\cdot,\bar{\mathcal{X}}(b),\omega,q)$ be the solution of (2.7) with input process $\bar{\mathcal{X}}(\cdot,b)$. Note that (4.51) in addition to (4.53) in the proof of Corollary 4.5 implies

$$E\sup_{0\le t\le T}\gamma_{\varpi}^4(\mathcal{X}_n(t\wedge\tau(c,b),b)-\bar{\mathcal{X}}(t\wedge\tau(c,b),b))\longrightarrow 0,\quad \text{as } n\longrightarrow\infty\,. \tag{4.59}$$

Therefore, by Theorem 2.1,

$$\left.\begin{array}{l}\sup_q E\sup_{0\le t\le T}\varrho^2(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q)-r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))\\ \le c_T\int_0^T E\gamma_{\varpi}^2(\bar{\mathcal{X}}(s\wedge\tau(c,b),b)-\mathcal{X}_{n-1}(s\wedge\tau(c,b),b))ds\\ \le c_T\sqrt{\int_0^T E\gamma_{\varpi}^4(\bar{\mathcal{X}}(s\wedge\tau(c,b),b)-\mathcal{X}_{n-1}(s\wedge\tau(c,b),b))ds}\longrightarrow 0,\quad \text{as } n\longrightarrow\infty\,.\end{array}\right\} \tag{4.60}$$

We set

$$\bar{\bar{\mathcal{X}}}(\cdot,b,\omega):=\int\delta_{r(\cdot,\bar{\mathcal{X}}(b),\omega,q)}\mathcal{X}_{0,b}(dq). \tag{4.61}$$

By Proposition 3.2

$$\bar{\bar{\mathcal{X}}}(\cdot,b,\omega)\in L_{p,\mathcal{F}}(C([0,\infty);\mathbf{M}_{\infty,\varpi}))\ \forall p\ge 1. \tag{4.62}$$

Proposition 4.7

For all positive rational numbers b and c

$$E\int_0^T\gamma_{\varpi}^4(\bar{\bar{\mathcal{X}}}(t\wedge\tau(c,b),b)-\bar{\mathcal{X}}(t\wedge\tau(c,b),b))dt=0. \tag{4.63}$$

Proof

By (4.59),

$$\begin{array}{l}E\int_0^T\gamma_{\varpi}^4(\bar{\bar{\mathcal{X}}}(t\wedge\tau(c,b),b)-\bar{\mathcal{X}}(t\wedge\tau(c,b),b))dt\\ =\lim_{n\to\infty}\int_0^T E\gamma_{\varpi}^4(\bar{\bar{\mathcal{X}}}(s\wedge\tau(c,b),b)-\mathcal{X}_{n-1}(s\wedge\tau(c,b),b))ds\\ =\lim_{n\to\infty}E\int_0^T\sup_{\|f\|_{L,\infty}\le 1}\times\\ \times[\int[f(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q))\varpi(r(t\wedge\tau,\bar{\mathcal{X}}(b),q))-f(r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))\varpi(r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))]\mathcal{X}_{0,b}(dq)]^4\\ \le\tilde{c}\lim_{n\to\infty}E\int_0^T\int\mathcal{X}_{0,b}(dq)\times\\ \times\varrho^4(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q)-r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))[\varpi(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q))+\varpi(r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))]\\ \quad(\text{as in } (4.48),(4.49)\ \text{ by Hölder's inequality}).\end{array}$$

By (5.1) and the boundedness of $\varpi(\cdot)$ Itô's formula in addition to (4.29) implies that $\varpi(r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),\omega,q))$ is uniformly integrable with respect to the product measure $P(d\omega)\otimes\mathcal{X}_{0,b}(dq)$, since $\mathcal{X}_{0,b}(dq)$ is $\sigma-$finite.[9] Therefore,

[9] Cf., Bauer (1968), Section 20, Cor. 20.5.

$\tilde{c}\lim_{n\to\infty} E\int_0^T \int \mathcal{X}_{0,b}(dq)\times$
$\times\varrho^4(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q) - r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))[\varpi(r(t\wedge\tau(c,b),\bar{\mathcal{X}}(b),q)) + \varpi(r_n(t\wedge\tau(c,b),\mathcal{X}_{n-1}(b),q))]$
$\longrightarrow 0,\quad$ as $n\longrightarrow\infty$ (by (4.52) and Lebesgue's dominated convergence theorem).

■

Since for $\mathcal{X}(0)\in L_{0,\mathcal{F}_0}(\mathbf{M}_{\varpi,\infty})$ we have $\mathcal{X}(\cdot)$, defined in (4.49), is in $L_{0,\mathcal{F}}(C([0,\infty);\mathbf{M}_{\infty,\varpi}))$, the solution of (2.7) with input process $\mathcal{X}(\cdot)$ and start q is well defined. Employing the measurable version of Part 3) of Theorem 2.1 we set

$$\left.\begin{array}{l}\sum_{j\in\mathbf{N}}\int\delta_{r(\cdot,\mathcal{X},\omega,q)}\mathcal{X}_{0,j}(\omega,dq)1_{\Omega_j\setminus\Omega_{j-1}}(\omega) = \int\delta_{r(\cdot,\mathcal{X},\omega,q)}\sum_{j\in\mathbf{N}}\mathcal{X}_{0,j}(\omega,dq)1_{\Omega_j\setminus\Omega_{j-1}}(\omega)\\ = \int\delta_{r(\cdot,\mathcal{X},\omega,q)}\mathcal{X}_0(\omega,dq).\end{array}\right\}\tag{4.64}$$

Therefore, Proposition 4.7 in addition to (4.50), (4.59) and (4.60) implies

Theorem 4.8

Assume (2.9). Then the measure valued process $\mathcal{X}(\cdot)$, defined by (4.57), is a solution of the quasi-linear SPDE (4.2) with initial condition $\mathcal{X}_0\in L_{0,\mathcal{F}_0}(\mathbf{M}_{\varpi,\infty})$. $\mathcal{X}(\cdot)\in L_{0,\mathcal{F}}(C([0,\infty);\mathbf{M}_{\infty,\varpi}))$ and has the flow representation

$$\mathcal{X}(\cdot,\omega) = \int\delta_{r(\cdot,\mathcal{X},\omega,q)}\mathcal{X}_0(\omega,dq).\tag{4.65}$$

■

Remark 4.9

(i) Kotelenez (2007a), Section 8, shows that, under additional smoothness conditions on the coefficients and the initial condition, the solution of (4.2) is unique.

(ii) Further, Kotelenez (2007b) obtains that, under additional smoothness conditions on the coefficients and the condition that $\mathcal{J}$ does not depend on the empirical distribution, the solution of (4.2) has a Stratonovich representation in the form of a first order stochastic transport equation:

$$d\mathcal{X} = -\bigtriangledown\cdot(\mathcal{X}F(\cdot,\mathcal{X},t))dt - \bigtriangledown\cdot(\mathcal{X}\int\mathcal{J}(\cdot,p,t))w(dp,\circ dt),\qquad \mathcal{X}(0)=\mathcal{X}_0.\tag{4.66}$$

Cf. also Kotelenez (2007a). The analogous statement obviously holds for the bilinear SPDE (4.1). ■

5 Appendix

Here, we derive some properties of ϖ. For $k,\ell = 1,..,d$ and f twice continuously differentiable we recall the notation

$$\partial_k f := \frac{\partial}{\partial r_k}f,\quad \partial^2_{k\ell}f := \frac{\partial^2}{\partial r_k\partial r_\ell}f.$$

Let $\beta > 0$ and consider $\varpi_\beta(r) := \varpi^\beta(r)$. We easily verify:

$$\left.\begin{array}{c}\sum_{k=1}^d |\partial_k \varpi_\beta(q)| + \sum_{k,\ell=1}^d |\partial^2_{k\ell}\varpi_\beta(q)| \le c\varpi_\beta(q),\\ |\varpi_\beta(r) - \varpi_\beta(q)| \le \gamma\beta\varrho(r,q)[\varpi_\beta(r) + \varpi_\beta(q)].\end{array}\right\} \tag{5.1}$$

Indeed, a simple calculation yields

$$\begin{array}{c}\partial_k \varpi(r) = -\gamma(1+|r|^2)^{-\gamma-1}2r_k\\ \partial^2_{k,\ell}\varpi(r) = -\gamma(-\gamma-1)(1+|r|^2)^{-\gamma-2}2r_k 2r_\ell - \gamma(1+|r|^2)^{-\gamma-1}2\delta_{k,\ell}.\end{array}$$

This implies the first inequality. The second inequality is obvious if $|r-q| \ge 1$. For the case $|r-q| < 1$ we may assume $|r| < |q|$ and $\beta = 1$. Then

$$\begin{array}{c}|\varpi(r) - \varpi(q)| = \varpi(r) - \varpi(q) = \int_{|r|}^{|q|} 2\gamma u(1+u^2)^{-(\gamma+1)}du\\ \le \int_{|r|}^{|q|}\gamma(1+u^2)^{-\gamma}du \quad (\text{as } 2u \le (1+u^2)\ )\\ \le \int_{|r|}^{|q|}\gamma(1+|r|^2)^{-\gamma}du \quad (\text{as } (1+u^2) \text{ is monotone decreasing for } u \ge 0\ )\\ = (|q|-|r|)\varpi(r) \le |r-q|\varpi(r) \le |r-q|[\varpi(r)+\varpi(q)].\end{array}$$

■

Further,

$$\varpi^{-1}(q)\varpi(r) = \left(\tfrac{1+|q|^2}{1+|r|^2}\right)^\gamma \le 2^\gamma(1+|r-q|^2)^\gamma. \tag{5.2}$$

(5.2) follows from

$$\begin{array}{c}\varpi^{-1}(q) = 1+|q|^2 = 1+|q-r+r|^2 \le 1+2|q-r|^2+2|r|^2\\ \le 2(1+|q-r|^2+|r|^2) \le 2(1+|r-q|^2)(1+|r|^2).\end{array}$$

To define the Wasserstein metric γ_f on $\mathbf{M}_f$ consider the space of all continuous Lipschitz functions f from $\mathbf{R}^d$ into $\mathbf{R}$. Denote this space by $C_L(\mathbf{R}^d;\mathbf{R})$. Further, $C_{L,\infty}(\mathbf{R}^d;\mathbf{R})$ is the space of all uniformly bounded Lipschitz functions f from $\mathbf{R}^d$ into $\mathbf{R}$. Abbreviate[10]

$$|||f||| := \sup_q |f(q)|; \quad \|f\|_L := \sup_{\{r\ne q, |r-q|\le 1\}} \tfrac{|f(r)-f(q)|}{\rho(r-q)}; \quad \|f\|_{L,\infty} := \|f\|_L \vee |||f|||. \tag{5.3}$$

For $\mu, \nu \in \mathbf{M}_f$ we define the Wasserstein metric on $\mathbf{M}_f$ by:

$$\gamma_f(\mu-\nu) := \sup_{\|f\|_{L,\infty}\le 1} \left|\int f(q)(\mu(dq)-\nu(dq))\right|. \tag{5.4}$$

We have the continuous inclusion:

$$(\mathbf{M}_f, \gamma_f) \subset (\mathbf{M}_{\infty,\varpi}, \gamma_\varpi).$$

[10]In the definition of the Lipschitz norm (15.37) we may, without loss of generality, restrict the quotient to $|r-q| \le 1$, since for values $|r-q| > 1$ the quotient is dominated by $2|||f|||$.

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