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Wave Motion 28 (1998) 333–352

WAVE MOTION

Propagation and localization of Rossby waves over random topography (two-layer model)

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Received 22 September 1997; received in revised form 31 March 1998; accepted 3 April 1998

Abstract

We study propagation of Rossby waves over randomly stratified bottom topography in a two-layer system. The problem is reduced to coupled stochastic wave equations in latitudinal variable with suitable boundary conditions. When the two-dimensional system is broken into its basic *barotropic* and *baroclinic* modes, each one is scattered by the medium, and generates the like-wise and the cross-wise components. We study statistics of the reflection and transmission coefficients for both types of generated modes, and show *localization* for the like-wise components, and *propagation* for the cross-wise components. The localization lengths and the transmission rates are estimated in terms of the basic parameters of the system and the correlation-function of topographic fluctuations. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Rossby waves are large scale, low frequency oscillations in the ocean and atmosphere, that are primarily responsible for the transfer of energy from the "large scale" sources to "small" (dissipation, absorption) scales, and thus for restoring the geostrophic balance. Two factors play an important role in their generation and propagation: the *beta-effect* (latitudinal variation of the local Coriolis parameter) and the *bottom profile*. The effect of topography on such waves depends largely on the relation between the characteristic wave length λ and the horizontal scale of topographic inhomogeneities l_h (see, e.g. [1]). In the practically important case $l_h \ll \lambda$ such a topography could sustain propagation of long waves with or without the β-effect. This fact is also used in the laboratory simulation of Rossby waves (see e.g. [2,3]).

Another important factor of the large-scale atmospheric and oceanic structure is its density stratification (cf. [4]). The simplest way to model such a structure is a single-layer (*barotropic*) system, where vertical motions are ignored (depth-averaged) to produce the effective horizontal dynamics. Such an approximation, however, is often too crude and limited to account for many significant phenomena related to density variations (buoyancy, convection, mixing, internal shear, etc.). The simplest way to incorporate some of those *baroclinic* features is via the two-layer approximation.

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PII: S0165-2125(98)00019-5

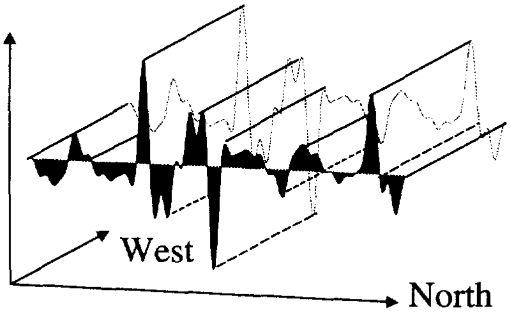

Fig. 1. Random topographic profile and its cross-section.

In this paper we shall study two-layer quasi-geostrophic (QGS) flows and the corresponding Rossby waves propagating over randomly stratified topography with latitudinal striation. One could visualize such a topography as a random array of (westward) parallel mountain ranges [1] (Fig. 1), and describe it by a single-variable "random" function $h(y)$, y-northward direction.

Complex topography is often modeled by periodic/quasi-periodic functions, or more general Fourier mode superpositions (cf. [5]). The "random" viewpoint (based on statistical sampling) could be justified on several grounds. On the one hand it could reflect the inherent uncertainty of the precise shape of h (particularly its smaller scales). On the other hand it could result from the dynamic evolution process, if the quantity of interest (flow field or wave) has sufficiently long temporal–spatial exposure to the medium (compared to its correlation scales), and thus "accumulated" large number of representative realizations. Then the ergodic principle is often invoked to justify "ensemble-averaging" and statistical sampling. Besides "statistical sampling and averaging" there are situations, where certain "coherent" features and phenomena reveal themselves in almost all realizations (of the process, field etc.) with probability 1. Wave localization by random (stratified) media belongs to such a class.

The general goal of statistical analysis is to uncover the mean-field characteristics and dynamics as well as their random fluctuations about mean, and to relate them to the basic characteristics of the medium. In the context of Rossby-waves over random topography such approach was developed in several papers, e.g. [6–10].

Most of them deal with a single-layer QGS-model and (latitudinally) stratified topography. In this case the (latitudinal) Rossby propagation could be reduced to a boundary-value Schrödinger problem with random potential, or the Helmholtz equation with random refractive index. The latter was extensively studied over the last few years in the context of *disordered media* ("transport" and "localization" phenomena in systems with random impurities), and the wave propagation (acoustic, electromagnetic) in randomly stratified media. We mention a few relevant works [11,12]; as well as monographs and papers [13,15,16] by Klyatskin, and paper [17].

Papers [9,10] applied some ideas and methods of the "random media" to a single-layer Rossby case. Their main conclusion was to show the *dynamic localization* of the latitudinal Rossby component for a typical realization of random topography. The relevant parameter called "localization length" was calculated in terms of spectral density of the random profile and the Rossby wave numbers.

Building on the earlier works by Klyatskin [9] and Klyatskin and Gurarie [10], Gryanik and Klyatskin [18] have recently developed a general framework for the study of "propagation and localization" for two-component systems, applicable to a wide range of wave motions.

Here we shall adopt the general method of [18] to study Rossby waves in the two-layer systems. The first preliminary attempt at such an analysis was made in [10]. There we observed some marked differences in the nature of the "two-layer" localization compared to a "single-layer" one, in particular, the localization length was shown to be much larger in the former case.

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[1] The exact East–West orientation is not essential, and could be replaced with any other unidirectional random profile aligned along the basic (background) β-plane.

Here we shall give more in-depth analysis of the problem. Clearly, the two-layer dynamics (propagation, scattering, etc.) becomes more involved compared to one-layer. We start by identifying two components of the system, named "barotropic" and "baroclinic", which propagate freely in the uniform (homogeneous) medium, but are coupled (scattered) by any amount of (topographic) inhomogeneity. In the scattering process each incident mode (of either type) generates a pair of the like-wise and cross-wise scattered components: "barotropic" ⇒ "barotropic + baroclinic"; "baroclinic" ⇒"barotropic + baroclinic".

The detailed statistical analysis of the scattering process reveals that the "like-generated" modes are still localized in space. We estimate their localization radii in terms of the basic parameters: relative depth of two layers, Rossby wave number, statistics of the random profile, the "baroclinicity" parameter, etc.

A new qualitative feature appears in the propagation of the "cross-generated" modes ("barotropic"– for incident "baroclinic", and "baroclinic"– for incident "barotropic"). It turns out that the cross-modes are not localized by the random medium (no matter how big is disorder!). We estimate the effective transmission rates, first in a finite-width (inhomogeneous) band, then in the infinite-space limit, and find them to be strictly positive. So unlike the one-layer model [8,10], two-layer Rossby waves could propagate (carry away energy) arbitrarily far northward and southward. However, in the process of propagation two modes switch their identity. Localization aside, another (possibly more practical) outcome of our analysis is to give a quantitative measure (transmission rates) of the attenuating effect of the "topographic disorder" on Rossby propagation.

Our paper exploits an overly simplified setup in terms of topography and density stratification, which stems from limitations of the basic "random media" tools and techniques adopted here. The latter are well developed in the stratified case (references above), but little, if any of this theory was carried over to a complete (2D or 3D) "disorder". That still eludes our efforts and poses a major challenge in the field of wave propagation. However, we could expect certain qualitative conclusions, like the failure of random obstacles to halt propagation, to remain true.

2. Quasi-geostrophic equations and Rossby waves for two-layer systems

We start with the basic two-layer QGS-system in a basin of variable depth (see Fig. (2) where the rigid lid was put on the top. It is described by a pair of coupled pde's (see e.g. [4])

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\Delta\psi_1-\alpha_1F(\psi_1-\psi_2)\right]+\beta\frac{\partial}{\partial x}\psi_1=J\left[\Delta\psi_1-\alpha_1F(\psi_1-\psi_2);\,\psi_1\right],\\&\frac{\partial}{\partial t}\left[\Delta\psi_2-\alpha_2F(\psi_2-\psi_1)\right]+\beta\frac{\partial}{\partial x}\psi_2=J\left[\Delta\psi_2-\alpha_2F(\psi_2-\psi_1)+f_0\alpha_2H;\,\psi_2\right].\end{aligned}\tag{1}$$

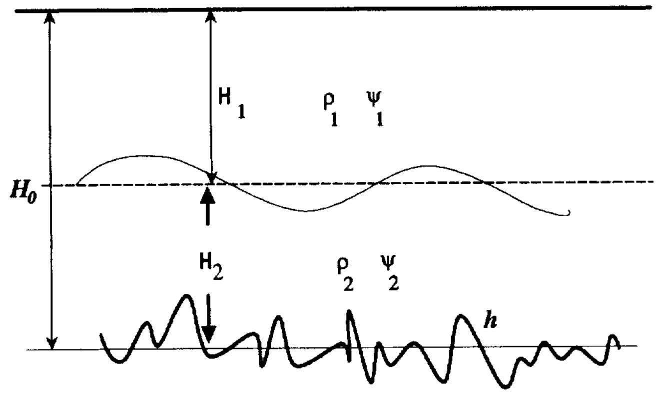

Fig. 2. Two-layer system with a rigid lid.

Here we used the following notations:

- ψ_1, ψ_2 – stream functions of upper and lower layers respectively;
- $\alpha_1 = 1/H_1, \alpha_2 = 1/H_2$ – reciprocals of the mean depths of two layers H_1 (top), H_2 (bottom);
- f_0 – the local (latitude-dependent) Coriolis parameter;
- β – its derivative in y ;
- $F = (f_0^2/g)(\delta\rho/\rho)$ combines local Coriolis, gravity acceleration, and density variation about its mean value $\rho_0 = (\rho_1 + \rho_2)/2$; $\delta\rho/\rho = (\rho_2 - \rho_1)/\rho_0$;
- $J(a; b) = ((\partial a/\partial x)(\partial b/\partial y) - (\partial b/\partial x)(\partial a/\partial y))$ – the standard Jacobian of two functions;
- $H = H(x, y)$ – a (variable) depth of the bottom profile in the Cartesian variables: x – westward, and y – northward.

We remark that parameter F has vertical (latitudinal) dependence and determines the horizontal (x, y)-scale of the baroclinic effects. For the Earth atmosphere at mid-latitudes, $F \sim 10^{-9}\ \mathrm{m}^{-1}$is sufficiently small.

Our model of the topographic medium has y-dependent cylindrical profile $H(y) = H_0 + h(y)$ (Figs. 1 and 2), where $H_0 = H_1 + H_2$ is the combined "mean" thickness of two layers, while h gives its randomly fluctuating component.

Rossby waves are linearized perturbations of the QGS equation (1) about special rest states (e.g. zonal currents). A case of interest to us is the rest state $\Psi_0 = 0$. The corresponding linearized Rossby waves are made of the Fourier harmonics in x, t of wave number κ and frequency ω, and have more complicated dependence in the latitudinal variable y.

We write two Rossby components as

$$\psi_1(y)\mathrm{e}^{-\mathrm{i}(\omega t+\kappa x)}, \qquad \psi_2(y)\mathrm{e}^{-\mathrm{i}(\omega t+\kappa x)}, \tag{2}$$

where signs $\omega > 0, \kappa > 0$, indicate the principal westward propagation. The y-dependent factors ψ_1, ψ_2 of (2) satisfy a linear differential system

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_1 + k^2\psi_1 - \alpha_1 F(\psi_1 - \psi_2) = 0, \\ &\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_2 + k^2\left[1 + \varepsilon(y)\right]\psi_2 + \alpha_2 F(\psi_1 - \psi_2) = 0. \end{aligned} \tag{3}$$

Two new parameters introduced in (3) are random scalar potential-function:

$$\varepsilon(y) = \frac{\kappa f_0}{H_2\omega k^2}\frac{\mathrm{d}}{\mathrm{d}y}h(y) \tag{4}$$

and the generalized eigenvalue (y-wave number)

$$k^2 = \kappa\left(\frac{\beta}{\omega} - \kappa\right) \tag{5}$$

depending on κ and ω.

Let us remark that a single-layer case could be obtained from (3) by making $F = 0, \phi_1 = 0$. Then (3) is reduced to the standard Helmholtz equation with refractive index ε

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi + k^2[1 + \varepsilon(y)]\psi = 0. \tag{6}$$

Function $\varepsilon(y)$ could be real or complex,

$$\varepsilon(y) = \varepsilon_1(y) + \mathrm{i}\gamma, \quad (0 < \gamma \ll 1), \tag{7}$$

its imaginary part describing wave attenuation by the medium.

The one-layer reduced Rossby model (6) was studied by Segupta et al. [8], by a combination of the analytical and numerical techniques. Their goal was to compute the Lyapunov exponent that characterizes the localization of Rossby modes in terms of the boundary-value problem for (6).

We shall adopt a somewhat different setup, following [9,10,18], based on the point-source problem for (6). Let us remark that "point source" here refers to the y-variable only, the x-dependence being periodic of the fixed wave-length $2\pi/\kappa$. Such Rossby modes would look similar to Kelvin–Helmholtz unstable modes in the context of shear-slip discontinuity. So one could think of such Rossby waves as generated by a discontinuous (slip) layer in the underlying (westward) zonal current.

In the rest of the paper we shall analyze solutions of (6), first as a dynamic (differential equation) problem, then we look at their statistical characteristics.

3. Dynamical formulation

3.1. Point source inside the layer

We assume topographic inhomogeneities to be localized in a finite width band $L_0 \le y \le L$, and to levels out outside this range (Fig. 3).

Let us consider a system of equations for the two-component (vector) Green function $\psi = (\psi_1, \psi_2)$

$$\begin{aligned} &\frac{d^2}{dy^2}\psi_1 + k^2\psi_1 - \alpha_1 F(\psi_1 - \psi_2) = -v_1\delta(y - y_0), \\ &\frac{d^2}{dy^2}\psi_2 + k^2[1 + \varepsilon(y)]\psi_2 + \alpha_2 F(\psi_1 - \psi_2) = -v_2\delta(y - y_0). \end{aligned} \tag{8}$$

Placing source in the first/second equation corresponds to the wave-generation in upper/lower layer, or a combined source, and $v_{1,2}$-indicate the strength of two components. Boundary conditions for problem (8) are the radiation conditions at $\pm\infty$ and continuity for wave-fields and their derivatives at the boundaries $y = L$ and $y = L_0$.

Introducing vector-function $\psi(y; y_0) = (\psi_1(y; y_0), \psi_2(y; y_0))$ and vector $\boldsymbol{v} = (v_1, v_2)$ we can rewrite system (8) in the vector form

$$\left[\frac{d^2}{dy^2} + A^2 + k^2\varepsilon(y)\Gamma\right]\psi(y; y_0) = -\boldsymbol{v}\delta(y - y_0), \tag{9}$$

with matrix-coefficients

$$A^2 = \begin{bmatrix} k^2 - \alpha_1 F & \alpha_1 F \\ \alpha_2 F & k^2 - \alpha_2 F \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

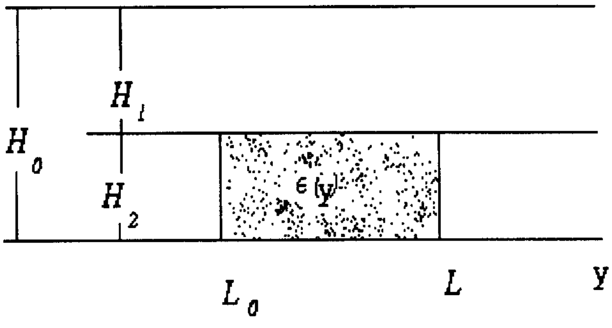

Fig. 3. Schematic geometry of the problem

By analogy with the optical propagation, matrix A plays the role of the constant (mean) component of the refraction index, while $\varepsilon\Gamma$ gives the randomly fluctuating one. From (10) one could easily compute A and its inverse

$$A = k\begin{bmatrix} \tilde{\alpha}_2 + \lambda\tilde{\alpha}_1 & (1-\lambda)\tilde{\alpha}_1 \\ (1-\lambda)\tilde{\alpha}_2 & \tilde{\alpha}_1 + \lambda\tilde{\alpha}_2 \end{bmatrix}, \qquad A^{-1} = \frac{1}{k\lambda}\begin{bmatrix} \tilde{\alpha}_1 + \lambda\tilde{\alpha}_2 & -(1-\lambda)\tilde{\alpha}_1 \\ -(1-\lambda)\tilde{\alpha}_2 & \tilde{\alpha}_2 + \lambda\tilde{\alpha}_1 \end{bmatrix}. \tag{11}$$

An important parameter that appears here

$$\lambda^2 = \left[1 - (\alpha_1 + \alpha_2)\frac{F}{k^2}\right] \tag{12}$$

encodes the baroclinic effects. We have also introduced two relative thickness parameters for the upper/lower layers

$$\tilde{\alpha}_1 = \alpha_1/(\alpha_1 + \alpha_2) = H_2/H_0, \qquad \tilde{\alpha}_2 = \alpha_2/(\alpha_1 + \alpha_2) = H_1/H_0,$$

so $\tilde{\alpha}_1 + \tilde{\alpha}_2 = 1$.

We shall assume the baroclinicity parameter $\lambda^2 > 0$, that is we consider only those modes, whose spatial scale along the y-axis does not exceed the internal Rossby scale $L_{\text{int}} = 1/[(\alpha_1 + \alpha_2)F]^{1/2}$. For the mid-latitude atmosphere L_{int} is of the order of 10^6 m – fairly large.

Next we introduce the fundamental matrix-solution Ψ of (9)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + A^2 + k^2\varepsilon(y)\Gamma\right]\Psi(y; y_0) = -E\delta(y - y_0), \tag{13}$$

and write vector-solution ψ in terms of Φ

$$\psi(y; y_0) = \Psi(y; y_0)\boldsymbol{v} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\psi_{11} + v_2\psi_{12} \\ v_1\psi_{21} + v_2\psi_{22} \end{bmatrix}. \tag{14}$$

Two columns of Φ correspond to the Rossby waves generated by sources $(v_1, 0)$ and $(0, v_2)$ in the upper and lower layers, respectively. Furthermore, fundamental matrix Φ satisfies the boundary conditions

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - \mathrm{i}A\right)\Psi(y; y_0)\bigg|_{y=L} = 0 \qquad \left(\frac{\mathrm{d}}{\mathrm{d}y} + \mathrm{i}A\right)\Psi(y; y_0)\bigg|_{y=L_0} = 0. \tag{15}$$

The problem could be simplified (cf. [9,10,18]) by placing the source at the boundary of the inhomogeneous region, $y_0 = L$. In this case, the "source-condition" becomes the "jump-condition" at y_0

$$\frac{\mathrm{d}}{\mathrm{d}y}\Psi(y; y_0)\bigg|_{y=y_0+0} - \frac{\mathrm{d}}{\mathrm{d}y}\Psi(y; y_0)\bigg|_{y=y_0-0} = -E,$$

and we reformulate (13) as the boundary-value problem

$$\begin{gathered} \left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + A^2 + k^2\varepsilon(y)\Gamma\right]\Psi(y; L) = 0, \\ \left(\frac{\mathrm{d}}{\mathrm{d}y} - \mathrm{i}A\right)\Psi(y; L)\bigg|_{y=L} = E; \qquad \left(\frac{\mathrm{d}}{\mathrm{d}y} + \mathrm{i}A\right)\Psi(y; L)\bigg|_{y=L_0} = 0. \end{gathered} \tag{16}$$

3.2. Generation of barotropic and baroclinic modes

To get a more physical view of the system we shall linearly transform the dependent variables in (16) to a new basis, that would make the homogeneous (constant) part of the "matrix refractive index" A diagonal. The corresponding components of Ψ will be called *baroclinic* and *barotropic* modes. The transformation is implemented by constant matrix

$$K = \begin{bmatrix} 1 & -1 \\ \tilde{\alpha}_2 & \tilde{\alpha}_1 \end{bmatrix} \tag{17}$$

that takes A into

$$B = KAK^{-1} = k \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$$

and transforms Φ into the fundamental matrix

$$U(y; L) = -2\mathrm{i}K\Phi(y; L)K^{-1}B. \tag{18}$$

The latter also solves the boundary-value problem:

$$\begin{gathered} \left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + B^2 + k^2\varepsilon(y)\tilde{\Gamma}\right] U(y; L) = 0, \\ \left(\frac{\mathrm{d}}{\mathrm{d}y} - \mathrm{i}B\right) U(y; L)\,|_{y=L} = -2\mathrm{i}B; \quad \left(\frac{\mathrm{d}}{\mathrm{d}y} + \mathrm{i}B\right) U(y; L)\,|_{y=L_0} = 0. \end{gathered} \tag{19}$$

Here the matrix-potential (refractive index) consists of the constant diagonal part B and the randomly fluctuating component

$$\tilde{\Gamma} = K\Gamma K^{-1} = \begin{bmatrix} \tilde{\alpha}_2 & -1 \\ -\tilde{\alpha}_1\tilde{\alpha}_2 & \tilde{\alpha}_1 \end{bmatrix}$$

The fundamental-matrix U becomes diagonal in the homogeneous medium, $\varepsilon = 0$, its entries U_{11} and U_{22} being the (uncoupled) *baroclinic* and *barotropic* modes. A non-constant topographic factor $\varepsilon\tilde{\Gamma}$, however, would couple two modes, so that the incident baroclinic wave would generate "baroclinic U_{11}" + "barotropic U_{21}", while the incident barotropic wave would generate "baroclinic U_{12}" + "barotropic U_{22}".

From system (19) one could easily estimate the magnitude of the induced barotropic component $U_{21} = \mathrm{O}(\delta)$, where

$$\delta = \lambda\tilde{\alpha}_1\tilde{\alpha}_2 = \lambda\frac{H_1H_2}{H_0^2} < \lambda/4, \tag{20}$$

a small parameter.[2] So one could write

$$U_{21} = \delta\tilde{U}_{21}, \tag{21}$$

where $\tilde{U}_{21}$ is of the same order as the other terms of matrix U.

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[2] Parameter δ depends on the relative depths of two layers and the "baroclinicity number" λ (12). In the typical geophysical setup two layers have widely disparate depth, so $\tilde{\alpha}_1\tilde{\alpha}_2 \ll 1$, and parameter δ is indeed small. For the atmosphere, lower layer $H_2 \ll H_1$-upper layer, so $\tilde{\alpha}_1 \ll 1$, $\tilde{\alpha}_2 \approx 1$, while the opposite holds in the ocean: $H_1 \ll H_2$. If two depth are comparable $H_1/H_2 \sim 1$, parameter δ would be small provided $\lambda \ll 1$. We recall that λ (12) is always between 0 and 1.

Next we shall introduce the scattering (reflection, transmission) matrices:

$$R(L) = U(L;L) - E, \qquad T(L) = U(L_0;L), \tag{22}$$

where E denotes 2×2 identity matrix. The matrix entries R_{ij} and T_{ij} represent its (complex) *reflection* and *transmission* coefficients for the like-wise components ($i = j$), and the cross-components (baroclinic $\rightleftharpoons$ barotropic) ($i \neq j$).

System (19) has two conserved integrals of motion

$$\tilde{\alpha}_1\tilde{\alpha}_2\left(U_{11}^*(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{11}(y) - U_{11}(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{11}^*(y)\right) + U_{21}^*(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{21}(y) - U_{21}(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{21}^*(y) = \text{Const.}$$

and

$$\tilde{\alpha}_1\tilde{\alpha}_2\left(U_{12}^*(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{12}(y) - U_{12}(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{12}^*(y)\right) + U_{22}^*(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{22}(y) - U_{22}(y)\frac{\mathrm{d}}{\mathrm{d}y}U_{22}^*(y) = \text{Const.},$$

which correspond to the energy flux conservation of two modes (see [9,10,18]).

One can rewrite the conservation laws in terms of the reflection and transmission coefficients, as

$$\delta(1 - |R_{11}|^2 - |T_{11}|^2) = |R_{21}|^2 + |T_{21}|^2, \qquad 1 - |R_{22}|^2 - |T_{22}|^2 = \delta(|R_{12}|^2 + |T_{12}|^2),$$

The latter could be renormalized via factoring out parameter δ, as in (21), to bring it to a symmetric form

$$1 - |R_{11}|^2 - |T_{11}|^2 = \delta(|\tilde{R}_{21}|^2 + |\tilde{T}_{21}|^2), \qquad 1 - |R_{22}|^2 - |T_{22}|^2 = \delta(|R_{12}|^2 + |T_{12}|^2), \tag{23}$$

where $R_{11} = \delta\tilde{R}_{21}$ and $T_{11} = \delta\tilde{T}_{21}$.

Let us point out that the *reflection–transmission coefficients* (22) bear the direct relevance to the *localization phenomena* of interest to us. Indeed, the localization occurs if the transmission coefficients T_{ij} tends to zero as the width of the inhomogeneous band $(L - L_0) \to \infty$.

The boundary value problem of type (19) was studied by Klyatskin [9], Klyatskin and Gurarie [10] and Gryanik and Klyatskin [18], using the *imbedding method* of [16] (see Appendix A.1). The idea is to view wave-fields $U_{ij}(y;L)$ as functions of the boundary parameter L, and transform the boundary-value problem (19) into the initial value problem with respect to L. Such an "initial value" reformulation makes the boundary-value problem amenable to statistical analysis developed in the context of random processes.

The resulting initial value problem has the form

$$\begin{aligned}
&\frac{\partial}{\partial L}U(y;L) = \mathrm{i}U(y;L)B + \frac{\mathrm{i}}{2}k^2\varepsilon(L)U(y;L)B^{-1}\tilde{\Gamma}U(L;L),\\
&U(y;L)|_{L=y} = U(y;y),\\
&\frac{\mathrm{d}}{\mathrm{d}L}U(L;L) = -2iB + i[U(L;L)B + BU(L;L)]\\
&\quad + \frac{\mathrm{i}}{2}k^2\varepsilon(L)U(L;L)B^{-1}\tilde{\Gamma}U(L;L), \qquad U(L;L)|_{L=L_0} = E.
\end{aligned} \tag{24}$$

Its second equation could be recast as the matrix Riccati equation for reflection-coefficients $R(L) = U(L;L) - E$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}L}R(L) = \mathrm{i}[R(L)B + BR(L)] + \frac{\mathrm{i}}{2}k^2\varepsilon(L)\,[E + R(L)]\,B^{-1}\tilde{\Gamma}[E + R(L)],\\
&R(L)|_{L=L_0} = 0.
\end{aligned} \tag{25}$$

Symmetric matrix R has only three independent entries R_{11}, R_{12} and R_{22}, as $R_{21} = \delta R_{12} = \tilde{R}_{21}$, that obey a differential system (25)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}L}R_{11} &= 2\mathrm{i}\lambda k R_{11} + \mathrm{i}\tilde{\varepsilon}(L)[\tilde{\alpha}_2(1+R_{11})^2 - 2\delta(1+R_{11})R_{12} + \delta\lambda\tilde{\alpha}_1 R_{12}^2], \\ \frac{\mathrm{d}}{\mathrm{d}L}R_{22} &= 2\mathrm{i}kR_{22} + \mathrm{i}\tilde{\varepsilon}(L)[\lambda\tilde{\alpha}_1(1+R_{22})^2 - 2\delta(1+R_{22})R_{12} + \delta\tilde{\alpha}_2 R_{12}^2], \\ \frac{\mathrm{d}}{\mathrm{d}L}R_{12} &= \mathrm{i}k(1+\lambda)R_{12} + \mathrm{i}\tilde{\varepsilon}(L)([\tilde{\alpha}_2(1+R_{11}) + \lambda\tilde{\alpha}_1(1+R_{22}) - \delta R_{12}]R_{12} \\ &\quad - (1+R_{11})(1+R_{22})). \end{aligned} \tag{26}$$

Here we introduced new random coefficient $\tilde{\varepsilon}$ by renormalizing ε (4).

$$\tilde{\varepsilon}(L) = \frac{k}{2\lambda}\varepsilon(L) \tag{27}$$

System (26) will serve a basis for the statistical analysis in the next section.

4. Statistical analysis

So far we made no specific assumptions about statistics of the random topographic factor $h(y)$, hence $\tilde{\varepsilon}(y)$. From now on $h(y)$ will be assumed a homogeneous, zero-mean, Gaussian field with the correlation-function and spectral density

$$B_h(y-y') = \langle h(y)h(y')\rangle, \qquad \Phi_h(q) = \int\limits_{-\infty}^{+\infty} \mathrm{d}\xi\, B_h(\xi)\mathrm{e}^{\mathrm{i}q\xi}. \tag{28}$$

We also assume it to have a finite correlation length l_h, i.e. $B_h(y) = B(y/l_h)$.

Then random coefficient $\tilde{\varepsilon}(y)$ (27) is also a homogeneous, zero-mean Gaussian random field with the auto-correlation and spectral functions

$$B_{\tilde{\varepsilon}}(\xi) = -\left(\frac{\kappa f_0}{2\lambda H_2\omega k}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}B_h(\xi), \qquad \Phi_{\tilde{\varepsilon}}(q) = \left(\frac{\kappa f_0}{2\lambda H_2\omega k}\right)^2 q^2\Phi_h(q), \tag{29}$$

its variance being

$$\sigma_{\tilde{\varepsilon}}^2 = B_{\tilde{\varepsilon}}(0) = \langle\tilde{\varepsilon}^2(y)\rangle = \frac{1}{2\pi}\left(\frac{\kappa f_0}{2\lambda H_2\omega k}\right)^2 \int\limits_{-\infty}^{+\infty} \mathrm{d}q\, q^2\Phi_h(q).$$

4.1. *Some results on the "single-layer" statistics*

In the single-layer case the reflection coefficient $R = \psi - 1$ satisfies the scalar Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}L}R_L = 2\mathrm{i}kR_L + \mathrm{i}\frac{k}{2}\tilde{\varepsilon}(L)(1+R_L)^2, \quad R_{L_0} = 0, \tag{30}$$

a consequence of (6) along with the boundary conditions (19).

The energy-flux conservation gives the identity

$$1-|R_L|^2=|T_L|^2. \tag{31}$$

The statistical analysis of R and T is based on the *diffusion approximation* method [13] and the resulting *Fokker–Planck* equation for the probability density function (p.d.f.) of the reflected intensity field

$$W=|R_L|^2. \tag{32}$$

The Fokker–Planck equation takes the form

$$\frac{\partial}{\partial L}P_L(W)=D\frac{\partial}{\partial W}\left[-(1-W)^2+\frac{\partial}{\partial W}W(1-W)^2\right]P_L(W). \tag{33}$$

with the effective diffusion coefficient

$$D=\left(\frac{\kappa f_0}{H\omega}\right)^2\Phi_h(2k), \tag{34}$$

computed explicitly in terms of the basic parameters: κ, f_0, H, ω and spectral density Φ_h. We remark that D has the dimension of the inverse distance.

The diffusion approximation is valid under certain assumptions on the *correlation radius* l_h of h vs. the "*effective diffusion rate*" D, namely,

$$Dl_h\ll 1. \tag{35}$$

In other words "long" Rossby wave do not feel random inhomogeneities on scales $\le l_h$ and propagate freely. In addition, derivation of (33) requires to "average" fast oscillations of the wave-field, which imposes an additional constraint

$$k/D\gg 1 \tag{36}$$

in terms of the principal latitudinal Rossby wave number k (5).

The diffusion approximation makes the problem statistically equivalent to a problem of the white-noise potential ε' of zero mean $\langle\varepsilon'(y)\rangle=0$, and correlation

$$B_{\varepsilon'}(\xi)=2\Phi_\varepsilon(2k)\,\delta(\xi)=2\left(\frac{2\kappa f_0}{H_0\omega k}\right)^2\Phi_h(2k)\delta(\xi).$$

Furthermore, if $kl_h\ll 1$, the correlation-function is approximated via

$$B_{\varepsilon'}(\xi)\approx 2\left(\frac{2\kappa f_0}{H_0\omega k}\right)^2\Phi_h(0)\,\delta(\xi),\qquad \Phi_h(0)\approx\sigma_h^2 l_h,$$

so the effective diffusion coefficient

$$D=\left(\frac{\kappa f_0}{H_0\omega}\right)^2\Phi_h(0) \tag{37}$$

depends only on the value of Φ_h at 0.

Of course, the opposite extreme $kl_h\gg 1$ ("short" Rossby waves relative to the medium variations) gives coefficient D which strongly depends on the specific form of spectral density Φ_h.

If topographic inhomogeneity is sufficiently wide-spread relative to the diffusion rate D, i.e. $D(L - L_0) \gg 1$, one could show that the Fokker–Planck p.d.f. (33) approaches the steady-state solution

$$P(W) = \delta(W - 1) \tag{38}$$

One obvious consequence of (38) is the unit value of the reflection coefficient (hence, zero transition, $|T|^2 = 0$) with probability one.

Furthermore, the wave-field intensity $I(y; L) = |U(y; L)|^2$ could be shown to be log-normal, which has several important consequences (see e.g. [16,17]) like constant mean intensity and exponentially increasing higher moments

$$\langle I^n(y; L)\rangle \sim e^{n(n-1)D(L-y)}, \quad n = 0, \pm 1, \pm 2, \dots . \tag{39}$$

Remark 1. From the physical standpoint log-normality means the existence of rare but large fluctuation in almost any realization of the process. The fluctuations occur over the general trend of exponential decay for *a typical realization*

$$I(y; L) = |U(y; L)|^2 \simeq 2 \exp\{-(L - y)/l_{\mathrm{loc}}\} \tag{40}$$

where l_{loc} gives the *localization length* of the model – a reciprocal of the diffusion rate D:

$$l_{\mathrm{loc}} = 1/D. \tag{41}$$

Here by *typical realization* we mean a deterministic curve, for which the random process spends on an average half the distance (time) above the curve and half below it. Let us remark that such *typical realization* could differ significantly from the ensemble average. In fact, one could estimate the wave-field intensity $I(y; L)$ in the entire range of variable $\xi = D(L - y)$, and show that the bound

$$I(y; L) < 8e^{-\xi/2}, \tag{42}$$

holds with probability 1/2. The type of behavior expressed by (40)–(42) is often associated with the localization phenomena in disordered systems.

So far we have discussed spatially localized inhomogeneity within a finite width band. Next we turn to the point source problem in the randomly stratified half-plane. Here all dynamical characteristics remain unchanged, however, the statistics of the mean and higher moments of intensity could change significantly. Furthermore, one needs a suitable regularization of the problem, since the wave intensity $I(y; y_0) = |U(y; y_0)|^2$ becomes infinite with probability 1. The infinitely large value of intensity means complete reflexivity by the random half-spaces: $y > y_0$ and $y < y_0$. Hence a steady-state could result if the energy is pumped into a "finite volume" between two halve-spaces over an infinitely long time interval. A regularization (e.g. small attenuation) could remove a such singularity and produce a finite answer.

Let us note that a dissipative part γ of the refraction coefficient (7) would modify the Fokker–Planck equation (33) by adding an extra transport term

$$\frac{\partial}{\partial L} P_L(W) = 2k\gamma \frac{\partial}{\partial W} W P_L(W) + D \frac{\partial}{\partial W}\left[-(1 - W)^2 + \frac{\partial}{\partial W} W (1 - W)^2\right] P_L(W) \tag{43}$$

(see e.g. [16]). In this case a steady state probability density exist

$$P(W) = \frac{2\beta}{(1 - W)^2} \exp\left(\frac{2\beta W}{1 - W}\right), \tag{44}$$

with parameter $\beta = k\gamma/D$ depending on γ.

We shall skip further details and draw some conclusions of the "single-layer analysis". Rossby waves are localized by randomly stratified topography. Some specific features of the localization include:

- log-normal distribution of fluctuations;
- exponentially decay of *typical realizations*;
- exponentially bounds for almost all realizations within any prescribed probability margin $p < 1$.

Now we shall turn to more complicated and qualitatively different two-layer case.

4.2. Statistical analysis of two-layer model

Our goal is to produce the analog of the Fokker–Planck (FP) equation (33) and study its solutions. The relevant probability density function (p.d.f.) P will depend on three reflection intensities

$$W_{ij} = \left|R_{ij}\right|^2, \qquad P = P_L(W_{11}, W_{22}, W_{12}). \tag{45}$$

The Fokker–Planck equation for P consists of a long and cumbersome formula (A.01) derived in Appendix A.2 (see also [18]). We will not bring the complete formula here, but discuss some important parameters, approximations and implications.

The FP equation (A.9) involves four diffusion coefficients, that play the role of a single diffusion rate D in the one-layer case. All four are expressed through the spectral function of $h(y)$,

$$\begin{aligned} D_1 &= \left[\frac{H_1 f_0 \kappa}{H_0 H_2 \omega}\right]^2 \Phi_h(2\lambda k), \qquad & D_2 &= \left[\frac{f_0 \kappa}{H_0 \omega}\right]^2 \Phi_h(2k), \\ D_3 &= \left[\frac{f_0 \kappa}{2\lambda H_2 \omega}\right]^2 \Phi_h(k(1+\lambda)), \qquad & D_4 &= \left[\frac{f_0 \kappa}{2\lambda H_2 \omega}\right]^2 (1-\lambda)^2 \Phi_h(k(1-\lambda)). \end{aligned} \tag{46}$$

In case of short-range random inhomogeneities, $\lambda k l_h \ll 1$, all four coefficients are expressed through the one "single-layer coefficient" D of the previous section (37),

$$D_1 = \left[\frac{H_1}{H_0}\right]^2 D, \quad D_2 = \left[\frac{H_2}{H_0}\right]^2 D, \quad D_3 = \frac{(1+\lambda)^2}{4\lambda^2} D, \quad D_4 = \frac{(1-\lambda)^2}{4\lambda^2} D \tag{47}$$

The derivation of the FP equation (A.9) is based on the diffusion approximation method and thus imposes some constraints, similar to a single-layer case. In particular, all four coefficients D_i must satisfy $D_i l_h \ll 1$. Also one needs to average the fast oscillating components of wave-fields, hence comes another constraint $k \gg D_i$ for all four D's.

To simplify FP equation (A.9) and produce some approximate solutions we observe that the dynamic equations (26), hence (A.9), involve small parameter δ (20). When all higher order terms $O(\delta^2)$ are dropped in (A.9), i.e. we disregard the secondary emission effects, the resulting equations decouple. So field-intensities W_{11}, W_{22} become statistically independent, and p.d.f. P factors into the product of one-variable functions.

The resulting factors $P_L(W_{11})$ and $P_L(W_{22})$ satisfy reduced FP equations:

$$\begin{aligned} \frac{\partial}{\partial L} P_L(W_{11}) &= D_1 \frac{\partial^2}{\partial W_{11}^2} W_{11}(1-W_{11})^2 P_L(W_{11}), \\ &\quad + \frac{\partial}{\partial W_{11}} [-D_1 (1-W_{11})^2 + 2\delta (D_3 + D_4) W_{11}] P_L(W_{11}), \\ \frac{\partial}{\partial L} P_L(W_{22}) &= D_2 \frac{\partial^2}{\partial W_{22}^2} W_{22}(1-W_{22})^2 P_L(W_{22}) \\ &\quad + \frac{\partial}{\partial W_{22}} [-D_2 (1-W_{22})^2 + 2\delta (D_3 + D_4) W_{22}] P_L(W_{22}). \end{aligned} \tag{48}$$

Each of two equations (48) resembles the "single-layer" case (33), but has an extra (first order) term

$$2\delta\,(D_3 + D_4)\,\frac{\partial}{\partial W} W P_L(W)$$

similar to a dissipative γ-term of (43). So the effect of the like-mode generation (baroclinic ⇒ baroclinic; barotropic ⇒ barotropic) by the incident wave is (statistical) dissipation: *real* refraction index $\tilde{\varepsilon}$ is replaced by the *complex* one, $\tilde{\varepsilon}(y) + i\delta(D_3 + D_4)$.

Passing to the half-space limit ($L_0 \to -\infty$), we get the steady state (L-independent) solutions of (48):

$$P(W_{11}) = \frac{2\gamma_1}{(1 - W_{11})^2} \exp\left(\frac{2\gamma_1 W_{11}}{1 - W_{11}}\right), \qquad P(W_{22}) = \frac{2\gamma_2}{(1 - W_{22})^2} \exp\left(\frac{2\gamma_2 W_{22}}{1 - W_{22}}\right), \tag{49}$$

with parameters

$$\gamma_1 = \delta\frac{D_3 + D_4}{D_1}, \quad \gamma_2 = \delta\frac{D_3 + D_4}{D_2}. \tag{50}$$

The latter measure the relative strength of the "decay" vs. "diffusion" terms in (48).

For small-scale topographic inhomogeneities parameters γ_i involve only relative thicknesses of two layers along with the (fixed) baroclinic wavelength λ (12):

$$\gamma_1 = \frac{H_2}{2\lambda H_1}(1 + \lambda^2), \qquad \gamma_2 = \frac{H_1}{2\lambda H_2}(1 + \lambda^2). \tag{51}$$

They are clearly independent of the detailed statistics of random inhomogeneities, and their product obeys

$$1 < \gamma_1\gamma_2 = \frac{1}{4}\left(\frac{1}{\lambda} + \lambda\right)^2 < \infty, \tag{52}$$

so small value of one of them would imply large value of the other.

Steady state p.d.f.'s (49) allow one to compute the statistics of the reflection coefficients. In particular, we could estimate their mean values

$$\langle W_{11}\rangle \approx 1 - 2\gamma_1 \ln(1/\gamma_1), \qquad \langle W_{22}\rangle \approx 1 - 2\gamma_2 \ln(1/\gamma_2) \tag{53}$$

for small $\gamma_i \ll 1$, and

$$\langle W_{11}\rangle \approx 1/2\gamma_1, \qquad \langle W_{22}\rangle \approx 1/2\gamma_2 \tag{54}$$

for large $\gamma_i \gg 1$.

As a consequence we get the transmission intensities $|T_{11}|^2$ and $|T_{22}|^2$ to be equal to zero with probability 1, in the asymptotic limit $L_0 \to -\infty$. Thus we have shown that the "like" scattered components of the incident barotropic/baroclinic waves are localized, and their localization lengths are determined by the *diffusion-coefficient* if "diffusion" dominates "decay", and by the *decay-coefficient* in the opposite case.

In the former case (dominant diffusion), i.e. $\gamma_1 \ll 1$ and $\gamma_2 \gg 1$, we get the localization lengths of two modes to be approximately equal

$$\begin{aligned} l^{(1)}_{\text{loc}} &= 1/D_1 = \left[\frac{H_0}{H_1}\right]^2 l_{\text{loc}} \quad \text{(barotropic)}, \\ l^{(2)}_{\text{loc}} &= 1/2\delta\,(D_3 + D_4) = \frac{\lambda H_0^2}{(1 + \lambda^2)H_1 H_2} l_{loc} \quad \text{(baroclinic)}, \end{aligned} \tag{55}$$

where l_{loc} denotes the single-layer localization length (41).

In second case (dominant decay), i.e. $\gamma_1 \gg 1$; $\gamma_2 \ll 1$, we get different length

$$l_{\mathrm{loc}}^{(1)} = 1/2\delta(D_3 + D_4) = \frac{\lambda H_0^2}{(1+\lambda^2)H_1 H_2} l_{\mathrm{loc}}, \qquad l_{\mathrm{loc}}^{(2)} = 1/D_2 = \left[\frac{H_0}{H_2}\right]^2 l_{\mathrm{loc}}. \tag{56}$$

The statistics of the "reflection" intensity W_{12} is more subtle, as it involves the cross-correlations of W_{12} with W_{11} and W_{22}. We shall use conservation equations (23) in the form

$$1 - \langle W_{11}\rangle - \delta\langle W_{21}\rangle = \delta\langle|\tilde{T}_{21}|^2\rangle, \qquad 1 - \langle W_{22}\rangle - \delta\,\langle W_{12}\rangle = \delta\langle|T_{12}|^2\rangle \tag{57}$$

to estimate the mean transmission intensities of generated waves.

If we substitute $T_1 = 1 - W_{11} - \delta W_{12}$ and $T_2 = 1 - W_{22} - \delta W_{12}$ in the Dirac delta, the resulting p.d.f.'s $P(T_j) = \delta(T_j)$ do not satisfy the FP equation (A.9). This means that the generated Rossby modes are not localized, otherwise, they would solve FP exactly, the same way as "single-layer" function $P(T) = \delta\,(T)$ with $T = 1 - W$, solves (33).

Next we exploit the symmetry of FP equation (A.9) with respect to indexes 1 and 2 to show $\langle W_{12}\rangle = \langle W_{12}\rangle$. Hence transmission intensities, $\langle|\tilde{T}_{21}|^2\rangle$, $\langle|T_{12}|^2\rangle$ are completely determined by the cross-terms of (57).

As above we consider two opposite extreme cases of "decay" vs. "diffusion":

- *Case:* $\gamma_1 \ll 1,\ \gamma_2 \gg 1$, or $(H_2/\lambda H_1 \ll 1)$. Here Eqs. (57) take the form

$$2\gamma_1 \ln(1/\gamma_1) = \delta\langle W_{12}\rangle + \delta\langle|\tilde{T}_{21}|^2\rangle, \qquad 1 - 1/2\gamma_2 = \delta\langle W_{12}\rangle + \delta\langle|T_{12}|^2\rangle, \tag{58}$$

Hence follow the "transmission" asymptotics

$$\langle|\tilde{T}_{21}|^2\rangle \sim \frac{2}{\delta}\gamma_1 \ln(1/\gamma_1), \qquad \langle|T_{12}|^2\rangle \sim \frac{1}{\delta}. \tag{59}$$

- *Case:* $\gamma_1 \gg 1,\ \gamma_2 \ll 1\quad (H_1/\lambda H_2 \ll 1)$. Here we get

$$\langle|\tilde{T}_{21}|^2\rangle \sim \frac{1}{\delta}, \qquad \langle|T_{12}|^2\rangle \sim \frac{2}{\delta}\gamma_2 \ln(1/\gamma_2). \tag{60}$$

Finally, we shall recast the transmission coefficients $\langle|T_{ij}|^2\rangle$ in the original basis of the "upper/lower layer" components (8) and (9), rather than the "baroclinic/barotropic" basis (17). We shall place source in the upper/lower layer at the boundary of inhomogeneous region, $y_0 = L$, compute the transmission intensities at the other boundary L_0 , then let $L_0 \to -\infty$. Two cases will be considered

1. Source placed in the lower layer ($v_1 = 0$, $v_2 = 1$). In this case

$$\begin{aligned} \langle|\psi_1(y)|^2\rangle|_{y=L_0} &= \langle|\psi_{12}|^2\rangle|_{y=L_0} = \frac{1}{4k^2}\frac{H_2^2}{H_0^2}\left(\frac{H_1^2}{H_0^2}\langle|\tilde{T}_{21}|^2\rangle + \frac{H_2^2}{H_0^2}\langle|T_{12}|^2\rangle\right), \\ \langle|\psi_2(y)|^2\rangle|_{y=L_0} &= \langle|\psi_{22}|^2\rangle|_{y=L_0} = \frac{1}{4k^2}\frac{H_1^2 H_2^2}{H_0^4}(\langle|\tilde{T}_{21}|^2\rangle + \langle|T_{12}|^2\rangle). \end{aligned} \tag{61}$$

2. Source placed in the upper layer ($v_1 = 1$, $v_2 = 0$). Here

$$\begin{aligned} \langle\phi_1(y)^2\rangle|_{y=L_0} &= \langle|\Phi_{11}|^2\rangle|_{y=L_0} = \frac{1}{4k^2}\frac{H_1^2 H_2^2}{H_0^4}(\langle|\tilde{T}_{21}|^2\rangle + \langle|T_{12}|^2\rangle), \\ \langle\phi_2(y)^2\rangle|_{y=L_0} &= \langle|\Psi_{21}|^2\rangle|_{y=L_0} = \frac{1}{4k^2}\frac{H_1^2}{H_0^2}\left(\frac{H_2^2}{H_0^2}\langle|\tilde{T}_{21}|^2\rangle + \frac{H_1^2}{H_0^2}\langle|T_{12}|^2\rangle\right). \end{aligned} \tag{62}$$

In either case (upper or lower-layer source) the transmission intensities of the generated Rossby modes are not zero, which implies the absence of localization. The exact values of these coefficients depend on the ration of two depth H_1/H_2, and the dimensional (baroclinicity) parameter λ (12). Furthermore, relations (61) and (62), along with asymptotics (59) and (60) and estimates (51) give some quantitative information of the propagation process.

5. Conclusions

We studied Rossby waves in the two-layered medium over latitudinally stratified random bottom topography. Viewed as a two-component system ("barotropic" plus "baroclinic" components) such Rossby wave is scattered by the medium, and the latter provides an effective coupling between two modes. Each incident wave gives rise (generates) both types of scattered modes: *baroclinic* $\Rightarrow$ *baroclinic* + *barotropic*; *barotropic* $\Rightarrow$ *barotropic* + *baroclinic*.

We have shown that the generated like-wise components of each mode ("generated baroclinic" for the "incident baroclinic", and "generated barotropic" for the "incident barotropic"), are localized the same way as in the single-layer case. Furthermore, we estimate their localization length in terms of the Rossby wave number, the relative depth of two layers and the statistics (correlation function, spectral density) of the random profile.

The new phenomena comes in the dynamics of "cross-wise" components of the generated waves. Those are no more localized. Their transmission coefficients are estimated and shown to remain finite (positive) as the width of the random topographic band tends to infinity. It remains finite in the limiting half-plane case. So no amount of topographic disorder could halt the propagation of Rossby waves in a two-layer system.

Our model of the Rossby generation and propagation is overly simplified, both in the choice of special (striated) topography, and the absence of zonal flows. However, it gives some interesting qualitative conclusions and indicates possible quantitative tools, that could be used in more realistic (atmospheric, oceanic) setup.

Acknowledgements

This work was supported by the CRDF grant no. RM1-272. V. Klyatskin and N. Gryanik were partly supported by the Russian Foundation for Basic Research (Projects: 95-05-14247, 96-05-65347 and 96-15-98527).

Appendix A

A.1. Imbedding method for wave problems

We shall give a brief outline of imbedding method [15], that allows to transform a 2-point boundary value problem on interval $[0, L]$ into another differential equation in boundary parameter L. A typical second order differential equation, or system, that appears in many models of wave propagation has the form

$$\begin{gathered}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\gamma(x)\frac{\mathrm{d}}{\mathrm{d}x}+K(x)\right)U(x)=0,\\ \left(\frac{\mathrm{d}}{\mathrm{d}x}+B\right)U(x)\Big|_{x=L}=D,\qquad \left(\frac{\mathrm{d}}{\mathrm{d}x}+C\right)U(x)\Big|_{x=0}=0\end{gathered}\tag{A.1}$$

Here B, C and D are constant (matrix) coefficients, while $\gamma(x)$, $K(x)$ vary in space. A standard approach to (A.1) is via reduction to a first order system, but that would double the system size and make it more cumbersome to analyze. So we shall work directly with (A.1).

We define the fundamental matrix $U(x;L)$ of (A.1) by inserting the identity matrix $E=\delta_{ij}$ in place of boundary value D, so any solution $U(x)=U(x;L)D$. Derivative of $U(x;L)$ the boundary parameter L obeys the same equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\gamma(x)\frac{\mathrm{d}}{\mathrm{d}x}+K(x)\right)\frac{\partial}{\partial L}U(x;L)=0. \tag{A.2}$$

Hence $\partial U(x;L)/\partial L$ itself is expressed through the fundamental matrix as

$$\frac{\partial}{\partial L}U(x;L)=U(x;L)\,\Lambda(L), \tag{A.3}$$

where $\Lambda(L)$ is the initial value of derivative $\partial U/\partial L$ at $x=L$. Alternatively (A.3) could be viewed as a differential equation in $L>x$ augmented with the initial condition

$$U(x;L)|_{L=x}=U(x;x). \tag{A.4}$$

Applying boundary operator $(\mathrm{d}/\mathrm{d}x+B)$ to (A.3) and setting $x=L$ we get the following expression for $\Lambda(L)$:

$$\Lambda(L)=D^{-1}(\gamma(L)-B)D+D^{-1}\{K(L)-\gamma(L)B+B^2\}U(L;L).$$

The boundary value $U(L;L)$ in turn obeys the relation

$$\frac{\mathrm{d}}{\mathrm{d}L}U(L;L)=\left.\left\{\frac{\partial}{\partial x}U(x;L)+\frac{\partial}{\partial L}U(x;L)\right\}\right|_{x=L}=D-BU(L;L)+U(L;L)\Lambda(L).$$

The latter could be also viewed as a differential equation with the initial condition at $L=0$

$$U(0;0)=(B-C)^{-1}D.$$

derived from (A.1).

A specific example of interest to us is the one-dimensional boundary value problem

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}+k^2(x)\right\}U(x)=0,$$

$$\left.\left(\frac{\mathrm{d}}{\mathrm{d}x}-\mathrm{i}k_0\right)U(x)\right|_{x=L}=-2ik_0,\qquad \left.\left(\frac{\mathrm{d}}{\mathrm{d}x}+\mathrm{i}k_0\right)U(x)\right|_{x=0}=0.$$

Here the imbedding equation for $U(x;L)$ takes the form

$$\frac{\partial}{\partial L}U(x;L)=U(x;L)\,\Lambda(L),\qquad U(x;L)|_{L=x}=U(x;x),$$

and $\Lambda(L)$ is computed explicitly

$$\Lambda(T)=\mathrm{i}k_0+\frac{\mathrm{i}}{2k_0}[k^2(L)-k_0^2]U(L;L).$$

Function $U(L;L)$ in turn solves the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}L}U(L;L)=2\mathrm{i}k_0[U(L;L)-1]+\frac{\mathrm{i}}{2k_0}[k^2(L)-k_0^2]U(L;L)$$

with initial condition $U(0;0)=1$.

A.2. Fokker–Planck equation for the reflected Rossby waves

Our goal here is to find the joint p.d.f. $P_L(W_{11}, W_{12}, W_{22})$ of the reflected wave-field intensities (45), in particular its evolution across the random topographic band, as a function of the width parameter L. For the sake of notation we relabel intensities W_{ij} by variables X, Y, Z

$$X(L) = W_{11} = |R_{11}|^2, \qquad Y(L) = W_{12} = |R_{12}|^2, \qquad Z(L) = W_{22} = |R_{22}|^2,$$

We start with the basic dynamic equations (26) for three components $\{R_{ij}\}$ of the reflection matrix and deduce the equations for their squares (intensities)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}L}X &= \mathrm{i}\tilde{\varepsilon}(L)[\tilde{\alpha}_2(R^*_{11} - R_{11})(1 - X) - 2\delta(R^*_{11}R_{12} - R_{11}R^*_{12})\\ &\quad - 2\delta(R_{12} - R^*_{12})X + \delta\lambda\tilde{\alpha}_1(R^*_{11}R^2_{12} - R_{11}R^{*^2}_{12})],\\ \frac{\mathrm{d}}{\mathrm{d}L}Z &= \mathrm{i}\tilde{\varepsilon}(L)[\tilde{\alpha}_1(1 - Z)(R^*_{22} - R_{22}) - 2\delta(R^*_{22}R_{12} - R_{22}R^*_{12})\\ &\quad - 2\delta Z(R_{12} - R^*_{12}) + \delta\lambda\tilde{\alpha}_2(R^*_{22}R^2_{12} - R_{22}R^{*^2}_{12})],\\ \frac{\mathrm{d}}{\mathrm{d}L}Y &= \mathrm{i}\tilde{\varepsilon}(L)[\tilde{\alpha}_2(R_{11} - R^*_{11})Y + \lambda\tilde{\alpha}_1(R_{22} - R^*_{22})Y - \delta(R_{12} - R^*_{12})Y\\ &\quad -(1 + R_{11})(1 + R_{22})R^*_{12} + (1 + R^*_{11})(1 + R^*_{22})R_{12}].\end{aligned} \tag{A.5}$$

The coefficients of system (A.5) depend on products of different R_{ij} and their conjugates, as such it is not closed and should be considered together with (26).

There is a standard procedure to compute FP density for any stochastic differential equation: $X' = V(X, t)$ with random "velocity" field V. One takes all solution-path $\{X(t)\}$ that connect two given points in time t, $X(0)$ and $X(t) = X$, for various realizations of V, and averages the associated Dirac-delta density over the ensemble $\{V\}$, so

$$P(X, t) = \langle\delta(X(t) - X)\rangle$$

To implement this procedure in our setup we introduce the delta-density on the W-space

$$\Phi_L(X, Z, Y) = \delta(X(L) - X)\delta(Z(L) - Z)\delta(Y(L) - Y),$$

and set

$$P_L(X, Y, Z) = \langle\Phi_L(X, Y, Z)\rangle_h,$$

subscript indicates the h-ensemble, as all coefficients in (26) and (A.5) are "functionals" of random h.

Delta-function Φ satisfies the Liouville equation

$$\begin{aligned}&\frac{\partial}{\partial L}\Phi_L(X, Y, Z)\\ &\quad = -i\tilde{\varepsilon}(L)\left\{\frac{\partial}{\partial X}[\tilde{\alpha}_2R^*_{11}(1 - X) - 2\delta R^*_{11}R_{12} - 2\delta XR_{12} + \delta\lambda\tilde{\alpha}_1R^*_{11}R^2_{12} - \text{c.c.}]\right.\\ &\qquad + \frac{\partial}{\partial Z}[\lambda\tilde{\alpha}_1R^*_{22}(1 - Z) - 2\delta R^*_{22}R_{12} - 2\delta ZR_{12} + \delta\tilde{\alpha}_2R^*_{22}R^2_{12} - \text{c.c.}]\\ &\qquad \left. + \frac{\partial}{\partial Y}[(\tilde{\alpha}_2R_{11} + \lambda\tilde{\alpha}_1R_{22} - \delta R_{12})Y - (1 + R_{11})(1 + R_{22})R^*_{12} - \text{c.c.}]\right\}\Phi_L(X, Y, Z).\end{aligned} \tag{A.6}$$

where c.c. refers to complex conjugated terms. We need to average (A.6) over random ensemble $\{\tilde{\varepsilon}(L)\}$ – related to $h(L)$ via (4) and (27).

Coefficients $\{R_{ij}\}$ are functionals of the Gaussian process $\tilde{\varepsilon}(L)$, so statistical means of (A.6) could be computed by the Furutsu–Novikov formalism [13,14,19,20]. The idea is split cross-correlations of $\tilde{\varepsilon}$ and its functionals $R[\tilde{\varepsilon}] = R_{ij}$. Given an arbitrary functional of the Gaussian zero-mean process $\tilde{\varepsilon}$, one has

$$\langle \tilde{\varepsilon}(L) R[\tilde{\varepsilon}(L)] \rangle = \int\limits_{L_0}^{L} \mathrm{d}\xi \, B_{\tilde{\varepsilon}}(L-\xi) \left\langle \frac{\delta}{\delta \tilde{\varepsilon}(\xi)} R[\tilde{\varepsilon}(L)] \right\rangle, \tag{A.7}$$

Formula (A.7) involves variational derivative $(\delta/\delta\tilde{\varepsilon}(\xi))R[\tilde{\varepsilon}(L)]$, that could be computed from the dynamic equations (26). The dominant contribution to integral (A.7) comes from the range $0 \le \xi \le l_h$ – the correlation length of h and $\tilde{\varepsilon}$. We recall that the wave dynamics could be considered deterministic on these scales, due to the "diffusion approximation" assumption (35) and (36) So the variational derivatives obey the first order initial-value ODEs

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}L} \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{11}(L) = 2\mathrm{i}\lambda k \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{11}(L), \\
&\left.\frac{\delta}{\mathrm{i}\delta\tilde{\varepsilon}(\xi)} R_{11}(L)\right|_{L=\xi} = [\tilde{\alpha}_2(1+R_{11})^2 - 2\delta(1+R_{11})R_{12} + \delta\lambda\tilde{\alpha}_1 R_{12}^2]_{L=\xi}, \\
&\frac{\mathrm{d}}{\mathrm{d}L} \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{22}(L) = 2\mathrm{i}k \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{22}(L), \\
&\left.\frac{\delta}{\mathrm{i}\delta\tilde{\varepsilon}(\xi)} R_{22}(L)\right|_{L=\xi} = [\lambda\tilde{\alpha}_1(1+R_{22})^2 - 2\delta(1+R_{22})R_{12} + \delta\tilde{\alpha}_2 R_{12}^2]_{L=\xi}, \\
&\frac{\mathrm{d}}{\mathrm{d}L} \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{12}(L) = \mathrm{i}k(1+\lambda) \frac{\delta}{\delta\tilde{\varepsilon}(\xi)} R_{12}(L), \\
&\left.\frac{\delta}{\mathrm{i}\delta\tilde{\varepsilon}(\xi)} R_{12}(L)\right|_{L=\xi} = [(\tilde{\alpha}_2(1+R_{11}) + \lambda\tilde{\alpha}_1(1+R_{22}) - \delta R_{12})R_{12} \\
&\qquad\qquad - (1+R_{11})(1+R_{22})]_{L=\xi},
\end{aligned}$$

easily solved. Taking into account explicit form of coefficient R_{ij} on sub-correlation scales ($\xi < l_h$)

$$R_{11}(\xi) = \mathrm{e}^{-2\mathrm{i}\lambda k(L-\xi)} R_{11}(L), \quad R_{22}(\xi) = \mathrm{e}^{-2\mathrm{i}k(L-\xi)} R_{22}(L), \quad R_{12}(\xi) = \mathrm{e}^{-\mathrm{i}k(1+\lambda)(L-\xi)} R_{12}(L),$$

we get variational derivatives expressed through functions R_{ij}

$$\begin{aligned}
\frac{\delta}{\mathrm{i}\delta\tilde{\varepsilon}(L-\xi)} R_{11}(L) &= \tilde{\alpha}_2(\mathrm{e}^{\mathrm{i}\lambda k\xi} + \mathrm{e}^{-\mathrm{i}\lambda k\xi} R_{11})^2 \\
&\quad - 2\delta(\mathrm{e}^{-\mathrm{i}k(1-\lambda)\xi} + \mathrm{e}^{-\mathrm{i}k(1+\lambda)\xi} R_{11})R_{12} + \delta\lambda\tilde{\alpha}_1 e^{-2ik\xi} R_{12}^2, \\
\frac{\delta}{\mathrm{i}\delta\tilde{\varepsilon}(L-\xi)} R_{22}(L) &= \lambda\tilde{\alpha}_1(\mathrm{e}^{\mathrm{i}k\xi} + \mathrm{e}^{-\mathrm{i}k\xi} R_{22})^2 \\
&\quad - 2\delta(\mathrm{e}^{\mathrm{i}k(1-\lambda)\xi} + \mathrm{e}^{-\mathrm{i}k(1+\lambda)\xi} R_{22})R_{12} + \delta\tilde{\alpha}_2 \mathrm{e}^{-2\mathrm{i}\lambda k\xi} R_{12}^2, \\
\frac{\delta}{i\delta\tilde{\varepsilon}(L-\xi)} R_{12}(L) &= -(\mathrm{e}^{\mathrm{i}\lambda k\xi} + \mathrm{e}^{-\mathrm{i}\lambda k\xi} R_{11})(\mathrm{e}^{\mathrm{i}k\xi} + \mathrm{e}^{-\mathrm{i}k\xi} R_{22}) \\
&\quad + \tilde{\alpha}_2(1 + \mathrm{e}^{-2\mathrm{i}\lambda k\xi} R_{11})R_{12} + \lambda\tilde{\alpha}_1(1 + \mathrm{e}^{-2\mathrm{i}k\xi} R_{22})R_{12} \\
&\quad - \delta_1 \mathrm{e}^{-\mathrm{i}k(1+\lambda)\xi} R_{12}^2.
\end{aligned} \tag{A.8}$$

When substituted in (A.6) and evaluated via (A.7) the resulting FP equation becomes

$$
\begin{aligned}
\frac{\partial}{\partial L} P_L(X,Z,Y) = \Bigg\{ & \frac{\partial}{\partial X}[-D_1(1-X)^2 - 4\delta^2 D_4 Y + 2\delta(D_3+D_4)X - \delta^2 D_2 Y - 4\delta^2 D_3 XY] \\
& + \frac{\partial}{\partial Z}[-D_2(1-Z)^2 - 4\delta^2 D_4 Y + 2\delta(D_3+D_4)Z - \delta^2 D_1 W_{12}^2 - 4\delta^2 D_3 ZY] \\
& + \frac{\partial}{\partial Y}[(D_1(1-X) + D_2(1-Z) + 2\delta(D_3+2D_4) - \delta^2 D_3 Y)Y \\
& \qquad - D_3(1+XZ) - D_4(X+Z)] \\
& + \frac{\partial^2}{\partial X} X[D_1(1-X)^2 + 4\delta^2 D_4 Y + 4\delta^2 D_3 XY + \delta^2 D_2 W_{12}^2] \\
& + \frac{\partial^2}{\partial Z} Z[D_2(1-Z)^2 + 4\delta^2 D_4 Y + 4\delta^2 D_3 ZY + \delta^2 D_1 W_{12}^2] \\
& + \frac{\partial^2}{\partial Y} Y[Y(D_1 X + D_2 Z + \delta^2 D_3 Y - 2\delta D_3) + D_3(1+XZ) \\
& + D_4(X+Z)] + 8\delta^2 D_3 \frac{\partial^2}{\partial X \partial Z} ZXY \\
& - 2\frac{\partial^2}{\partial X \partial Y} XY[D_1(1-X) + 2\delta(D_3+D_4) - 2\delta^2 D_2 Y] \\
& - 2\frac{\partial^2}{\partial Z \partial Y} ZY[D_2(1-Z) + 2\delta(D_3+D_4) \\
& - 2\delta^2 D_2 Y] \Bigg\} P_L(X,Z,Y),
\end{aligned}
\tag{A.9}
$$

It involves four diffusion coefficients

$$
D_1 = 2\tilde{\alpha}_2^2 \int_0^\infty \mathrm{d}\xi\, B_{\tilde{\varepsilon}}(\xi)\cos(2\lambda k\xi), \qquad D_2 = 2(\lambda\tilde{\alpha}_1)^2 \int_0^\infty \mathrm{d}\xi\, B_{\tilde{\varepsilon}}(\xi)\cos(2k\xi),
$$

$$
D_3 = 2\int_0^\infty \mathrm{d}\xi\, B_{\tilde{\varepsilon}}(\xi)\cos(k(1+\lambda)\xi), \qquad D_4 = 2\int_0^\infty \mathrm{d}\xi\, B_{\tilde{\varepsilon}}(\xi)\cos(k(1-\lambda)\xi),
$$

expressed through the correlation function of h. Hence follows the spectral function representation of D_i given in (46).

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