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Equilibrium states for quasigeostrophic flows with random topography

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Abstract

We study the 2-D quasigeostrophic flows in the presence of complex bottom topography. The meaning of randomness of the bottom topography is discussed and the proper model is offered. In the zeroth order approximation this model yields a quasigeostrophic equation with random Gaussian profile. We study the Gaussian stationary ensembles and derive explicit relations between the correlation functions and spectral densities of solution fields (velocity/stream function) and the topography. Statistic ensembles play important role in the 2-D hydrodynamic and were subject of numerous studies (see Kraichnan and Montgomery (1980; Isichenko (1992)and references there). Our results are consistent with the earlier works (Kraichnan et al.), but the method is different. We work directly with the moment equations and utilize their symmetries, rather than the Gibbsian energy/enstrophy ensembles on the phase space. We hope the scope of our technique could be extended to other cases, particularly a nonzero base current and (turbulent) viscocity dissipation.

Keywords: Quasigeostrophic flows; Random topography

1. Introduction

The quasigeostrophic approximation (QGS) describes large scale dynamics of incompressible fluid flows propagating over bottom topography subject to Coriolis forces. Its stream function $\psi(\boldsymbol{r}; t), \boldsymbol{r} = (x; y)$ obeys the following equation (see [1]):

$$\frac{\partial}{\partial t}\Delta\psi = J(\Delta\psi + h; \psi), \tag{1}$$

where $\boldsymbol{u} = (-\partial\psi/\partial y; \partial\psi/\partial x)$ denotes the velocity field, $J(\psi; \zeta) = (\partial\psi/\partial x)(\partial\zeta/\partial y) - (\partial\psi/\partial y)(\partial\zeta/\partial x)$ the standard Jacobian of two functions and $\zeta = \Delta\psi + h$ the potential vorticity. The topographic factor h combines the effect of the 'depth-variation over mean depth' $\delta H/H_0$, the local Coriolis parameter f_0 and the latitudinal change of the Coriolis parameter, so called β-factor

$$h = f_0\frac{\delta H}{H_0} + \beta y.$$

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Special case $h = 0$, $\beta = 0$ corresponds to the classical inviscid Euler 2-D fluid. We are interested in statistical solutions of (1), particularly the effects of complex topography (modeled by random function h) on stream-field ψ.

Equilibrium statistical ensembles in 2-D hydrodynamics were studied extensively starting with the pioneering work of Kraichnan [2] (see also [3–15]). The standard approaches involve the proper phase-space representation of the Euler/QGS-flow (e.g. point-vortex approximation, eigenmode expansion etc.) and the micro-canonical or canonical (Gibbs) ensembles based on the conserved integrals: energy, enstrophy, etc. Among various Gibbsian equilibria the Gaussian ones (based on energy-enstrophy) play particularly important role (see [4,6,7,10–12]). Here we shall adopt a different approach.

To introduce the proper models and equations let us first to elaborate on the 'random' nature of topography in QGS (1) and the resulting 'stochastization' of the flow. In dynamical system theory statistical averaging appears in several contexts. One natural setup involves a deterministic system $\dot{\psi} = F(\psi)$ on a 'large' (infinite) phase-space $\mathcal{P}$, where averaging results from *ergodicity* of the flow on joint level surfaces of its conserved integrals. A typical trajectory of such system would sample different (allowed) portions of the phase-space $\mathcal{P}$ with different frequencies, that would eventually accumulate to a continuous (microcanonical or canonical) Gibbs distribution on $\mathcal{P}$. So 'time averages' accumulate to the 'phase-space average' in this setup.

Another context involves systems with random parameters (coefficients): $\dot{\psi} = F(\psi; h)$, for instance stochastically driven dynamic systems with random forcing: $\dot{\psi} = F(\psi) + h$, or diffusion models of passive tracers in turbulent velocity fields. In such cases the role of the Gibbs ensemble is played by the 'Fokker–Planck'-type (stationary or nonstationary) distribution.

Randomness of topography reflects its geometric complexity. One possible manifestation of complexity could be multiscale (multifractal) structure of h exemplified by 'random' Fourier expansion

$$h = \sum h_m \cos k_m x$$

(random coefficients $\{h_m\}$), illustrated in Figs. 1 and 2 (see [13] for a detailed exposition of 'statistical topography'). Other forms of complex geometry could be produced by a particular shape or pattern randomly replicated/distorted over the plane.

In either case nonlinear interactions between various scales (wave numbers) of ψ and h are expected to bring about the stochastization of ψ via formation of the large–small-scale eddies and the transfer of energy/enstrophy among scales. The statistics of such equilibria $\{\psi\}$ (means, correlations, higher moments, the related power spectra etc.) could then be linked to statistics of the media h and/or fluxes of the conserved quantities.

There remains however the basic issue as to what extent could one model 'complex' geometry by random coefficient h in dynamic equations, like (1). Indeed, topography per se does not provide requisite 'sources' to sustain fluxes in the cascade scenario. Also 'complex' (even multiscale) function h could not be strictly speaking viewed 'random', as in any 'local' space region it is fixed. So the stream-field ψ could effectively interact only with a 'particular realization' of h.

One natural setup that brings about 'random sampling' of h by ψ and the ensuing stochastization arises in the context of a current propagating over complex bottom profile. In the simplest case we take a parallel flow $\Psi_0(\boldsymbol{r}) = V^{\perp} \cdot \boldsymbol{r}$ of constant velocity V. The mean field Ψ_0 solves only the mean (assumed to be zero: $\langle h \rangle = 0$) topographic QGS (1) and produces a (small-scale) perturbation 'eddy' field ψ. So the total stream-field is the sum $\Psi_0 + \psi$. As perturbation ψ is carried over by the mean current it could samples different 'local realizations' of random topography $\{h_t(\boldsymbol{r}) = h(\boldsymbol{r} - Vt) : t\}$. In such case the ensemble averaging of ψ results from *ergodicity* of h relative to space translations $\{h(\boldsymbol{r} - tV) : t > 0\}$. In other words the stochastization is brought about by multiple interactions of ψ with 'local realizations' of h along the mean flow Ψ_0, assumed to represent the entire h-ensemble [1].

[1] Such ergodicity assumption could be justified in some cases, for instance for Gaussian stationary random fields h.

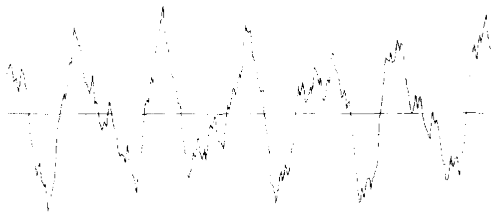

Fig. 1. Multiscale random topographic shape.

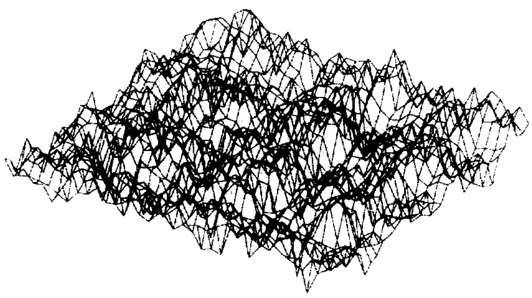

Fig. 2. Random Gaussian topography h.

Based on such observations we shall consider a simple parallel flow $\Psi_0(\boldsymbol{r}) = Vy$ perturbed by small-scale stream-field $\psi(\boldsymbol{r}, t)$ propagating over random h, and interacting with h according to (1). We are interested in the dynamics and statistics of the perturbed field $\psi(\boldsymbol{r}, t)$. The latter obeys the stochastic differential equation

$$(\partial_t + V\partial_x)\,\Delta\psi = J(\Delta\psi + h; \psi) - Vh_x.$$

Passing to a moving coordinate frame $r \to r - Vt$ we can rewrite the latter as

$$\partial_t \Delta\psi = J\left(\Delta\psi + \tilde{h}; \psi\right) - V\tilde{h}_x, \tag{2}$$

where $\tilde{h}$ indicates the time-dependent (moving) topography $\tilde{h} = h(\boldsymbol{r} - Vt)$. Notice that topography appears now in two places, as random coefficient inside the J-term and the random stirring force in the r.h.s.

$$F(\boldsymbol{r}, t) = Vh_x(\boldsymbol{r} - Vt) \tag{3}$$

similar to a wind-share. The statistics of $\tilde{h}$ and F are simply related to those of h, in particular both have zero means and their space–time correlation scales depend on two parameters: the correlation length l of h and the mean (current) velocity V. Precisely,

$$\left\langle F(\boldsymbol{r}, t)\, F(\boldsymbol{r}', t')\right\rangle = \sum_{i,j=1,2} v_i v_j \partial_{ij}^2 B_h\left(\boldsymbol{r} - \boldsymbol{r}' - V(t - t')\right). \tag{4}$$

where B_h is the correlation function of h. Assuming h to be homogeneous and isotropic random field we get

$$B_h(\boldsymbol{r} - \boldsymbol{r}') = \left\langle h(\boldsymbol{r})\, h(\boldsymbol{r}')\right\rangle = B_h\left(|\boldsymbol{r} - \boldsymbol{r}'|/l\right)$$

From (4) one deduces the time correlation scale of the 'forcing' term F to be

$$l_T \approx l/V.$$

So two distinct regimes appear in this setup.

(1) Short time-correlation $l/V \ll 1$ corresponds to large current velocity compared to the correlation length of h. Here one could apply the δ-type (white noise) diffusion approximation and produce a suitable Fokker–Planck (Novikov) equation for the distribution density $P_t = \cdots$ of ψ (dots indicate the proper transport and diffusion terms). Stationary Fokker–Planck solution could be used (in principle) to compute the correlation function

$$B_\psi\left(\boldsymbol{r},\boldsymbol{r}'\right) = \int \psi\left(\boldsymbol{r}\right)\psi\left(\boldsymbol{r}'\right) P\left(\psi\right)\,\mathrm{d}\psi$$

higher moments and the related power spectra. This work is currently in progress.

(2) Long time-correlation $l/V \gg 1$. This regime corresponds to a relatively slow current.

Our primarily interest in this paper will be the second case. Here the problem acquires a small parameter $\epsilon = V/l$, and has dimensionless form

$$\partial_t \Delta\psi = J\left(\Delta\psi + h;\psi\right) - \epsilon h_x \tag{5}$$

with h representing the 'moving' topography $h\left(\boldsymbol{r} - \epsilon t\right)$ of $\mathcal{O}\left(1\right)$ – correlation radius.

Eqs. (2) and (5) are nonlinear and involve rather complicated and implicit dependence of ψ on random parameter h. A direct approach to statistics of ψ entails the chain of coupled moment equations. The standard Gaussian closure would allow one to reduce the infinite system to its second and third moments (two-point and three-point correlators), and eventually produce a *kinetic-type* equation for correlators B_ψ; $B_{\psi h}$; B_h (similar to the weak turbulence theory [14]). The derivation of the proper kinetic equation and its analysis is fairly complicated, and so far we had only partial progress along these lines.

Let us remark that the proposed dimensionless form (5) offers another possible approach based on the perturbation expansion

$$\psi \approx \psi_0 + \epsilon\psi_1 + \cdots \tag{6}$$

In this paper we make the first step in such analysis and concentrate on its zeroth order term, i.e. solution of the equation

$$\partial_t \Delta\psi = J\left(\Delta\psi + h;\psi\right) \tag{7}$$

with Gaussian homogeneous isotropic field $h = h\left(\boldsymbol{r}\right)$. Solutions of (7) need not be Gaussian in general. It turned out however, that (7) does have stationary equilibrium solutions ψ with zero mean $\langle\psi\rangle = 0$. Furthermore, its correlation functions B_ψ ;$B_{\psi h}$ are simply (linearly!) related to correlator B_h.

The reason for simple 'linear' Gaussian solutions of highly nonlinear and complicated moment equations depends on special features of (7) and the corresponding kinetic equation. The nonlinear Jacobian bracket of (7) yields a quadratic (collision) term that could be factored into the product of two linear terms. Those naturally lead to linear differential relations between correlators B_ψ; $B_{\psi h}$; B_h. The final answer could be expressed in terms of spectral densities of ψ and h:

$$E_\psi\left(k\right) = \int \mathrm{d}\boldsymbol{r}\mathrm{e}^{\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{k}} B_\psi\left(\boldsymbol{r}\right) = \frac{C_1}{k^2 + k_0^2} + \frac{\left(\beta k^2\right)^2 E_h\left(k\right)}{\left(\alpha + \beta k^2\right)^2}. \tag{8}$$

Not surprisingly our solution (8) coincides with the well-known results based on the 'Gibbsian' approach ([4,7,10]), where parameters α, β represent the Kraichnan's energy–enstrophy temperatures ([4]). Our method however is different from the standard approaches as it does not require a priori the Gibbsian form on the limiting stationary distribution, nor the knowledge of the QGS-conserved integrals (an open problem in hydrodynamics, cf. [18]). Besides Kraichnan's solution (8) our analysis reveals another solution with the δ-type 'coherent' spectrum: $E(k) = C\,\delta(k - k_0)$. Its meaning and significance are not clear at the moment, a better understanding could be gleaned from the detailed perturbation analysis.

Solution (8) has several physical drawbacks, in particular divergent energy-term $C_1\left(k^2 + k_0^2\right)^{-1}$, due to the absence of viscous dissipation in the model. Adding dissipation to (7) is expected to provide the needed correction, but it changes significantly, the form of the kinetic equation and precludes the 'factorization trick'. [2] Viscous case along with the general perturbation method and the kinetic equation approach are part of our ongoing study still in progress. We expected to produce a variety of solutions in both stationary and non-stationary cases and find the requisite corrections to the energy spectrum (8).

In this paper we shall not pursue either of two methods in depth. Rather for the purpose of exposition we use the direct 'correlator-splitting' approach to stationary statistical solutions of (7). It was originally developed in the Euler case $h = 0$ in [5]. Here we elaborate it further to include topography and sketch some applications to the two-layered baroclinic flows and the turbulent diffusion problem.

2. Correlation function of stationary QGS-ensembles

We consider homogeneous and isotropic random fields h and random initial state ψ_0 with zero mean $\langle h(\boldsymbol{r})\rangle = \langle\psi\rangle = 0$ and the correlation function

$$B_h\left(|\boldsymbol{r} - \boldsymbol{r}'|\right) = \left\langle h(\boldsymbol{r})\, h\left(\boldsymbol{r}'\right)\right\rangle .$$

Here and throughout the paper angular brackets $\langle\cdots\rangle$ are used for statistical averaging over random ensembles. Our goal is to derive the equations for correlators B_ψ; $B_{\psi h}$ in terms of B_h. In what follows it would be more convenient to use the *structure function*

$$D_\psi\left(\boldsymbol{r} - \boldsymbol{r}'; t\right) = \left\langle\left[\psi(\boldsymbol{r}; t) - \psi\left(\boldsymbol{r}'; t\right)\right]^2\right\rangle = 2\left[B_\psi(0; t) - B_\psi\left(\boldsymbol{r} - \boldsymbol{r}'; t\right)\right],$$

rather than correlator $B_\psi\left(\boldsymbol{r}, \boldsymbol{r}'; t\right)$. Let us remark that all 2-point correlators B_ψ; B_h and the cross-term $B_{\psi h}$ depend only on the difference $\boldsymbol{r} - \boldsymbol{r}'$ – a consequence of homogeneity (translational symmetry) of the problem.

We start with the triple correlator

$$\langle\Delta\psi\,\Delta\psi\,\Delta\psi\rangle = \langle\Delta\psi(\boldsymbol{r}_1; t)\,\Delta\psi(\boldsymbol{r}_2; t)\,\Delta\psi(\boldsymbol{r}_3; t)\rangle ,$$

and similarly defined $\langle\Delta\psi\, h\, h\rangle$, and write the evolution (moment) equation for it

$$\frac{\partial}{\partial t}\langle\Delta\psi\,\Delta\psi\,\Delta\psi\rangle = \left\langle\Delta\dot\psi\,\Delta\psi\,\Delta\psi\right\rangle + \left\langle\Delta\psi\,\Delta\dot\psi\,\Delta\psi\right\rangle + \left\langle\Delta\psi\,\Delta\psi\,\Delta\dot\psi\right\rangle . \tag{9}$$

Clearly, the stationary regime gives 0 in the l.h.s. of (9). Next we replace derivative $\Delta\dot\psi$ by the quadratic form $J(\Delta\psi + h; \psi)$ according to the QGS-evolution (1), and expand the resulting quartic correlators the r.h.s. of (9) through the standard products of the 2-point correlators $\{\langle\psi\rangle\langle\psi\rangle\}$ or $\{\langle\psi\rangle\langle\psi h\rangle\}$ summed over all pair partitions (Gaussian closure). The resulting cumbersome expression of $\left\langle\Delta\dot\psi\,\Delta\psi\,\Delta\psi\right\rangle + \cdots$ combines the derivatives and products of the 2-point correlators of ψ and h.

[2] To compound the matters the dissipation term would itself depend on ψ through its effective turbulent-viscosity coefficient (its small-scale spectrum) that results from stochastization of the flow.

To simplify it and bring to a manageable form we shall introduce the following two-variable function X made of the products of derivatives of $D_\psi(q)$ and the cross-correlator $B_{\psi h}(q)$

$$X(p;q) = \frac{1}{pq}\frac{\partial^2}{\partial p \partial q}\{[\Delta_p^2 D_\psi(p)\ \Delta_q D_\psi(q) - \Delta_q^2 D_\psi(q)\ \Delta_p D_\psi(p)]$$
$$-2[\Delta_p B_{\psi h}(p)\,\Delta_q D_\psi(q) - \Delta_p D_\psi(p)\,\Delta_q B_{\psi h}(q)]\}. \tag{10}$$

Here vectors $\boldsymbol{p} = (\boldsymbol{x};\boldsymbol{y})$; $\boldsymbol{q} = \cdots$ in $\mathbb{R}^2$ have norms $p = |\boldsymbol{p}|$; $q = |\boldsymbol{q}|$, and Δ_p; Δ_q denotes the standard (radial) Laplacians in $\boldsymbol{p}$; $\boldsymbol{q}$

$$\Delta_q = \frac{\partial^2}{\partial q^2} + \frac{1}{q}\frac{\partial}{\partial q}.$$

The quartic correlator equation (9) will be shown in Appendix A to produce a simple functional relation for function X. Namely,

$$X(q_1;q_2) + X(q_2;q_3) + X(q_3;q_1) = 0 \tag{11}$$

for any triplets of coordinates $q_1 = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$; $q_2 = |\boldsymbol{r}_2 - \boldsymbol{r}_3|$; $q_3 = |\boldsymbol{r}_3 - \boldsymbol{r}_1|$.

We proceed next by observing that function $X(p;q)$ is antisymmetric in $p;q$

$$X(p;q) = -X(q;p).$$

Hence the general solution of (11) could be expressed through a single one-variable function $A(q)$ as

$$X(q_1;q_2) = A(q_1) - A(q_2). \tag{12}$$

Such representation (12) will allow to separate variables $p;q$ in (10) and to reduce it to a system of ordinary different equations relating B_h ; D_ψ ; $B_{\psi h}$. Indeed, multiplying (12) by $q_1 q_2$ and applying the fourth order differential operation $\partial^4/\partial q_1^2 \partial q_2^2$ we get a nonlinear separable pde in variables q_1; q_2

$$\frac{\partial^6}{\partial q_1^3 \partial q_2^3}\{\Delta_1^2[D_\psi(q_1)]\,\Delta_2[D_\psi(q_2)] - \Delta_2^2[D_\psi(q_2)]\,\Delta_1[D_\psi(q_1)]$$
$$-2\Delta_1[B_{\psi h}(q_1)]\,\Delta_2[D_\psi(q_2)] - 2\Delta_1[D_\psi(q_1)]\,\Delta_2[B_{\psi h}(q_2)]\} = 0. \tag{13}$$

Here Δ_1; Δ_2 abbreviate the radial Laplacians in q_1; q_2 respectively. Variables q_1; q_2 are easily separated in (13) and the latter is reduced to an ordinary differential equation

$$(\Delta_q + \lambda)\,\Delta_q D_\psi - 2\Delta_q B_{\psi h} = 0 \tag{14}$$

for unknown functions $D_\psi(q)$ and $B_{\psi h}(q)$. The separation procedure introduces a dimensional constant λ (inverse square length) related to the 'energy–enstrophy temperatures' of [4].

2.1. Euler case

To proceed further let us first review the Euler case $h = 0$ studied in [5]. Here Eq. (14) is simplified to

$$(\Delta_q + \lambda)\Delta_q[D_\psi(q)] = 0. \tag{15}$$

There are two classes of solutions (15), corresponding to positive and negative values $\lambda = k_0^2 > 0$ and $\lambda = -k_0^2 < 0$, those are related to the negative and positive 'temperature' of the Kraichnan's energy/enstrophy Gibbs ensembles.

In the former case $\lambda > 0$ the fourth order differential equation (15) is reduced to the second order one

$$\Delta_q \left[D_\psi (q)\right] = C\ J_0 (k_0 q),$$

with the first kind Bessel function $J_0(z)$ in the r.h.s. Hence follows the structure function and the appropriate spectral density

$$E(k) = E_0 \delta (k - k_0).$$

As we mentioned earlier the δ-type spectral density implies the higher level correlation (coherent structure) of the random field ψ, reflected in the slow (polynomial) fall-off of J_0.

In the second case $\lambda = -k_0^2 < 0$ Eq. (15) is reduced to

$$\Delta_q \left[D_\psi (q)\right] = -C_1 K_0 (k_0 q) \tag{16}$$

with the third kind Bessel K_0 and certain dimensional parameters k_0 and C_1. The corresponding spectral density becomes ([2–4])

$$E(k) = E_0/(k^2 + k_0^2). \tag{17}$$

To compute the structure function $D_\psi (q)$ we integrate (16) subject to the boundary conditions: $D_\psi (0) = 0$; $D'_\psi (0) = 0$, and find

$$D_\psi (q) = C\left[K_0 (k_0 q) + \ln\left(\tfrac{1}{2} k_0 q\right) + \gamma\right] \tag{18}$$

with the Euler constant γ. Notice that (18) gives the log-divergent solution at ∞, to get a bounded one we interchange two differential operations in (15) and solve it in the form

$$(\Delta_q + \lambda)\left[D_\psi (q)\right] = 0.$$

Let us remark that the 2-D stationary Gaussian ensembles for incompressible fluids could be either localized (δ-type) or decaying at ∞ (17), whereas the 3-D flows allow only the white noise (δ-correlated) solutions (see e.g. [15–17]).

2.2. *General topographic case*

Next we shall turn to the general topographic case (14). We look for solutions of (14) in the class of bounded functions $D_\psi (q)$, $B_{\psi h} (q)$. Eq. (14) could be recast in the form

$$\left(\Delta_q + \lambda\right)\left[D_\psi (q)\right] - 2B_{\psi h} (q) = 0,$$

or using the correlator B_ψ in place of D_ψ

$$\left(\Delta_q + \lambda\right)\left[B_\psi (q)\right] + B_{\psi h} (q) = 0. \tag{19}$$

Eq. (19) is not closed as it couples B_ψ to a new (yet undetermined) cross-correlation function $B_{\psi h}$. To determine the latter we take another 3-point correlator $\langle \Delta\psi\ h\ h\rangle$ and let it evolve according to (1). The stationary third moment equation

$$\frac{\partial}{\partial t} \langle \Delta\psi (\boldsymbol{r}_1; t)\ h(\boldsymbol{r}_2)\ h(\boldsymbol{r}_3)\rangle = 0$$

can be analyzed as above (via Gaussian splitting of the 4-point correlators) to get

$$\frac{1}{q_1 q_3}\frac{\partial^2}{\partial q_1 \partial q_3}\left\{\Delta_1\left[B_{\psi h}(q_1)\right]B_{\psi h}(q_3) - \Delta_3\left[B_{\psi h}(q_3)\right]B_{\psi h}(q_1)\right.$$
$$\left. + B_h(q_1)\,B_{\psi h}(q_3) - B_h(q_3)\,B_{\psi h}(q_1)\right\} = 0$$

analogous to (10) and (11).Then separation of variables $\{q_1; q_3\}$ yields an equation similar to (19)

$$\left(\Delta_q + \lambda_1\right)\left[B_{\psi h}(q)\right] + B_h(q) = 0 \tag{20}$$

with yet another separation constant λ_1. The combined system of two equations (19) and (20)

$$\left(\Delta_q + \lambda\right)\left[B_{\psi}(q)\right] + B_{\psi h}(q) = 0, \qquad \left(\Delta_q + \lambda_1\right)\left[B_{\psi h}(q)\right] + B_h(q) = 0 \tag{21}$$

is already closed, and gives the requisite relations among three correlators $B_{\psi h}$; B_ψ; B_h

$$B_\psi = \left(\Delta_q + \lambda\right)^{-1} B_{\psi h} = \left(\Delta_q + \lambda\right)^{-1}\left(\Delta_q + \lambda_1\right)^{-1} B_h.$$

One can formally solve (21) for $B_{\psi h}$;B_ψ with arbitrary values of parameters λ; λ_1. However, such solutions need not describe the Gaussian equilibrium ensemble $\{\psi; h\}$. Indeed, the latter will be shown to require the consistency conditions $\lambda = \lambda_1$, i.e.

$$B_\psi = \left(\Delta_q + \lambda\right)^{-2} B_h$$

2.3. Consistency condition and existence of Gaussian equilibria

We shall use the characteristic functional of random fields $(\psi; h)$

$$\Phi_t[v;\kappa] = \left\langle \exp\left\{ \mathrm{i}\int \mathrm{d}^2\boldsymbol{r}\,\left[\psi(\boldsymbol{r};t)v(\boldsymbol{r}) + h(\boldsymbol{r})\kappa(\boldsymbol{r})\right]\right\}\right\rangle$$
$$= \left\langle \exp \mathrm{i}\left\{(\psi|v) + (h|\kappa)\right\}\right\rangle.$$

Here $(f|g)$ indicates the standard L^2-inner product on functions over $\mathbb{R}^2$.Clearly, Φ_t contains all spatial statistics of fields $\{\psi(\boldsymbol{r};t); h(\boldsymbol{r})\}$ at time t. Furthermore, dynamic evolution (1) for ψ implies a linear evolution equation in variational derivatives for Φ_t known as the *Hopf equation* [19] (see also [15–17])

$$\frac{\partial}{\partial t}\Phi_t[v;\kappa] = -\mathrm{i}\left(v\left|\Delta^{-1}J\left(\Delta\frac{\delta}{\delta v} + \frac{\delta}{\delta\kappa};\frac{\delta}{\delta v}\right)\right.\right)\Phi_t[v;\kappa].$$

The limiting (equilibrium) state $\Phi_\infty = \lim_{t\to\infty}\Phi_t$ has $(\partial/\partial t)\Phi_\infty = 0$, i.e.

$$\left(\Delta^{-1}v\left|J\left(\Delta\frac{\delta}{\delta v} + \frac{\delta}{\delta\kappa};\frac{\delta}{\delta v}\right)\right.\right)\Phi_\infty[v;\kappa] = 0. \tag{22}$$

The characteristic functional Φ_∞ of the Gaussian field $(\psi; h)$ involves the standard quadratic/bilinear forms of $(v; \kappa)$ with coefficients given by the above B-correlators. Namely

$$\Phi_\infty[v;\kappa] = \exp\left\{-\tfrac{1}{2}\left[\left(v\left|B_\psi\right|v\right) + 2\left(v\mid B_{\psi h}\mid\kappa\right) + \left(\kappa\mid B_h\mid\kappa\right)\right]\right\}. \tag{23}$$

Here each bracket $(\cdots|B|\cdots)$ is viewed as an integral (convolution) operator applied to a suitable pair of functions $\{v; v\}$; $\{v; \kappa\}$; $\{\kappa; \kappa\}$, e.g.

$$(v \mid B_{\psi h} \mid \kappa) = \int\int \mathrm{d}^2\boldsymbol{r}\,\mathrm{d}^2\boldsymbol{r}'\, B(\boldsymbol{r}-\boldsymbol{r}')\; v(\boldsymbol{r})\, \kappa(\boldsymbol{r}').$$

Substitution of (23) in (22) yields after the proper use of variational derivatives

$$\left(\Delta^{-1}v\,|J(F;G)\right)\Phi_\infty - \left\langle\Delta^{-1}v\right|J(\Delta\psi + h;\psi)\rangle\,\Phi_\infty = 0 \tag{24}$$

with functional coefficients

$$F = \Delta B_\psi[v] + B_{\psi h}[v] + \Delta B_{\psi h}[\kappa] + B_h[\kappa], \qquad G = B_{\psi h}[v] + B_h[\kappa]. \tag{25}$$

Here operators $B\cdots[v]$; $B\cdots[\kappa]$ in (25) could be understood as integral kernels $B(\boldsymbol{r}-\boldsymbol{r}')$ applied to test functions v; κ

$$B[v](\boldsymbol{r}) = \int B(\boldsymbol{r}-\boldsymbol{r}')v(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'.$$

We observe that the second term $\langle\Delta^{-1}v|\,J(\Delta\psi + h;\psi)\rangle$ of (24) vanishes identically due to the statistical homogeneity and incompressibility of the problem (see Appendix A). Now it remains to substitute the correlator equations (21)

$$\Delta\left[B_\psi\right] + B_{\psi h} = -\lambda B_\psi, \qquad \Delta\left[B_{\psi h}\right] + B_h = -\lambda_1 B_{\psi h}$$

in (24) to get

$$\left(\Delta^{-1}v\left|J\left(\Delta\frac{\delta}{\delta v} + \frac{\delta}{\delta\kappa}; \frac{\delta}{\delta v}\right)\right)[\Phi_\infty] = (\lambda_1 - \lambda)\left(\Delta^{-1}v\,|J\left(B_\psi[v]; B_{\psi h}[\kappa]\right)\right)\Phi_\infty. \tag{26}$$

In order for the l.h.s. to vanish for an arbitrary choice of $\{v;\kappa\}$ (the stationary condition) two separation constants (21) must be equal

$$\lambda = \lambda_1.$$

That is precisely the consistency condition for the stationary Gaussian ensemble with correlators (21) to exist. Clearly, such ensemble would allow only for a single length scale determined by the separation constant λ.

We complete our section on the Gaussian QGS equilibria with the following remarks.

Remark 1. The correlator-function of the random solution $\{\psi\}$ of (1) was obtained by augmenting the 'flat bottom' solution of the homogeneous problem (19) and (20), with a particular inhomogeneous solution given by $B_h(q)$. Namely,

$$B_{\psi h}(q) = \left(\Delta_q + \lambda\right)^{-1}[B_h(q)], \qquad B_\psi(q) = B^0_\psi(q) + \left(\Delta_q + \lambda\right)^{-2}[B_h(q)], \tag{27}$$

where B^0_ψ is the flat (unperturbed) Bessel-type correlator J_0 or K_0.

Remark 2. The canonical 'Gibbsian' approach yields formulae (27), (17) and (18)), and the related spectral densities

$$E_\psi(k) = \left\langle\left|\hat\psi(k)\right|^2\right\rangle = \frac{k^2}{\alpha + \beta k^2} + \frac{\left(\beta k^2\right)^2 E_h(k)}{\left(\alpha + \beta k^2\right)^2} \tag{28}$$

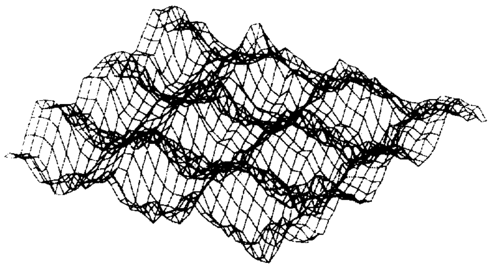

Fig. 3. The steam-field ψ of h.

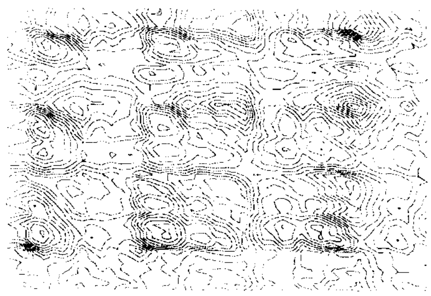

Fig. 4. The steam-lines of random field ψ.

and

$$\left\langle \hat{\psi}^* (k)\, \hat{h} (k) \right\rangle = \frac{k^2}{\alpha + \beta k^2} + \frac{\left(\beta k^2\right)^2 E_h (k)}{\left(\alpha + \beta k^2\right)^2} \tag{29}$$

via Fourier transform. Here $\hat{\psi} (k) : \hat{h} (k)$ denote the Fourier coefficients of ψ; h and E_ψ; E_h – spectral densities of ψ and h respectively. Indeed, Gibbs equilibrium

$$\mathrm{d}\mu (\psi) = Z \exp \{-\alpha E - \beta Q\}\, \mathrm{d}\psi$$

based on two quadratic functional of ψ: energy $E = \frac{1}{2} \int \Delta\psi\,\psi$ and enstrophy $Q = \frac{1}{2} \int \Delta\psi\,\Delta\psi$ yields the correlation

$$B_\psi = \left(\alpha \Delta + \beta \Delta^2\right)^{-1} B_h [\delta]$$

whence follows (28) and (29).

Figs. (2–4) illustrate random bottom topography h, the corresponding steady-state stream-field

$$\psi (\boldsymbol{r}) = (\alpha \Delta + \beta)^{-1} [h]$$

and its flow-lines.

3. Some extensions of the main result

Steady-state solutions ψ of the deterministic QGS-equation (1) are well known to satisfy

$$\Delta\psi + h = f(\psi),$$

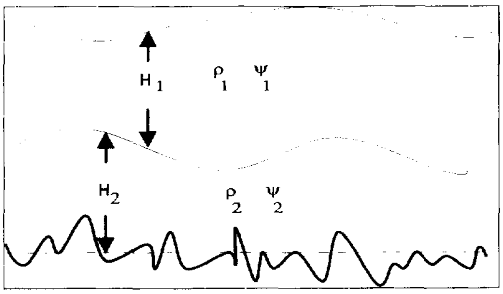

Fig. 5. Schematic view of the two-layered flow.

which in the simplest Fofonoff (linear) case [20] becomes

$$\Delta\psi + h = -\lambda\psi. \tag{30}$$

If one formally views (30) as the stochastic equation with random coefficient h one could easily compute the 2-point correlators for $(\psi;\psi)$ and $(\psi;h)$ and get the same system of differential relations (21) for them. In this sense the problem of Gaussian stationary ensembles is statistically equivalent to the stationary Fofonoff solutions (30). This conclusion draws on the well-known fact about Gaussian Gibbs ensemble (with deterministic factor h) [8]: the mean field $\langle\psi\rangle$ of any such ensemble solves the stationary Fofonoff equation (30).

3.1. Two-layer QGS flows

The basic quasigeostrophic equation (1) describes a single layered fluid and does not take into account the baroclinic effects. The latter could be approximately modeled by the two layered model, i.e. a coupled system of QGS-equations [1] (see Fig. 5)

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\Delta\psi_1 - \alpha_1 F(\psi_1 - \psi_2)\right] = J\left(\Delta\psi_1 - \alpha_1 F(\psi_1 - \psi_2);\psi_1\right),\\
&\frac{\partial}{\partial t}\left[\Delta\psi_2 - \alpha_2 F(\psi_2 - \psi_1)\right] = J\left(\Delta\psi_2 - \alpha_2 F(\psi_2 - \psi_1) + f_0\alpha_2 H;\psi_2\right)
\end{aligned} \tag{31}$$

where parameters $\alpha_1 = 1/H_1; \alpha_2 = 1/H_2$ denote inverse layer thicknesses; H–relative bottom profile $\rho_2; \rho_1$ – layer densities; f_0 – local Coriolis parameter and coefficient

$$F = \frac{f_0^2}{2g}\frac{\rho_1+\rho_2}{\rho_2-\rho_1} \quad (\rho_2 > \rho_1).$$

System (31) in the absence of topographic factor was considered in ([7]). One can use the methods of Section 2 to show that the Gaussian equilibrium for (31) is statistically equivalent to the steady-state Fofonoff solution of (31)

$$\Delta\psi_1 - \alpha_1 F(\psi_1 - \psi_2) = -\lambda_1\psi_1, \qquad \Delta\psi_2 - \alpha_2 F(\psi_2 - \psi_1) + f_0\alpha_2 H = -\lambda_2\psi_2 \tag{32}$$

Such equilibria are characterized by two 'eigenvalue' parameters $\lambda_1; \lambda_2$, i.e. the system has two length scales. Eqs. (32) can be written in the operator-matrix form

$$\begin{bmatrix} \Delta - \alpha_1 F + \lambda_1 & \alpha_1 F \\ \alpha_2 F & \Delta - \alpha_2 F + \lambda_2 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{bmatrix} 0 \\ f_0\alpha_2 H\end{bmatrix}.$$

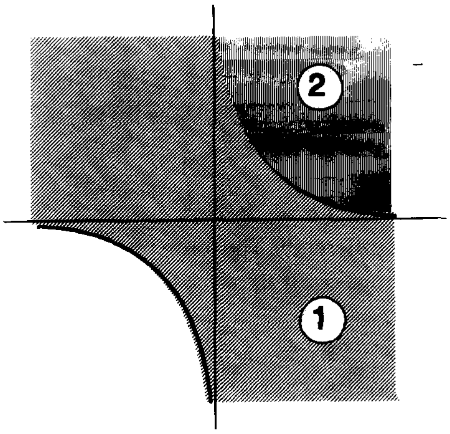

Fig. 6. Two regions of the coherent structure in the $\lambda\lambda$ – parameter plane: Region 1 corresponds to the single-scale coherent structure, region 2 to the double-scale one.

or in the equivalent scalar form

$$\hat{L}[\psi_1] = \alpha_1\alpha_2 F f_0 H, \qquad \hat{L}[\psi_2] = -\alpha_2 f_0 (\Delta + \lambda_1 - \alpha_1 F) H$$

with operator

$$\hat{L} = \Delta^2 + [\lambda_1 + \lambda_2 - F(\alpha_1 + \alpha_2)]\Delta + \{\lambda_1\lambda_2 - F(\lambda_2\alpha_1 + \lambda_1\alpha_2)\}.$$

Operator $\hat{L}$ can be factored in the product

$$(\Delta + \mu_1)(\Delta + \mu_2), \tag{33}$$

whose characteristic roots are expressed through the shifted eigenvalue parameters: $\tilde{\lambda}_{1,2} = \lambda_{1,2} - F\alpha_{1,2}$,

$$\mu_{1,2} = \tfrac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2) \pm \sqrt{\tfrac{1}{4}\left(\tilde{\lambda}_1 - \tilde{\lambda}_2\right)^2 + \alpha_1\alpha_2 F^2}.$$

The coherent structure (delta-type spectral density) corresponds to the positive characteristic roots (33). The resulting range of parameters $\tilde{\lambda}$ that give positive roots include (Fig. 6)

- $\tilde{\lambda}_1\tilde{\lambda}_2 < \alpha_1\alpha_2 F^2$ – single positive root,
- $\tilde{\lambda}_1\tilde{\lambda}_2 > \alpha_1\alpha_2 F^2$ and $\tilde{\lambda}_1 + \tilde{\lambda}_2 > 0$ – two positive roots.

We observe that the two-layered flows allows coherent (δ-correlated) structures with two length scales (region above the upper branch of the hyperbola $\tilde{\lambda}_1\tilde{\lambda}_2 = \alpha_1\alpha_2 F^2$).

3.2. Passive tracer transfer

Another application of our technique is to the passive transfer by random incompressible velocity fields. In the simplest one-layer case the basic QGS-flow (1) is supplemented by the tracer-density equation

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\frac{\partial\theta}{\partial\boldsymbol{r}} = 0; \quad \theta(\boldsymbol{r};0) = \theta_0(\boldsymbol{r}),$$

or in the stream-function formulation

$$\frac{\partial\theta}{\partial t} + J(\theta;\psi) = 0. \tag{34}$$

The initial state θ_0 is assumed to be a homogeneous random field with constant mean $\langle\theta_0(\boldsymbol{r})\rangle = \theta_0$. Clearly, the mean will remain constant $\langle\theta_0(\boldsymbol{r};t)\rangle = \theta_0$ for any time t. It is convenient to replace the positive tracer density θ by its log that takes on both positive and negative values $\theta \to \chi = \log\theta/\theta_0$. We consider fluctuations of χ about its mean value $\chi_0 = \langle\log\theta/\theta_0\rangle$, i.e. $\tilde{\chi} = \chi - \chi_0$. The evolution equation for $\tilde{\chi}$ retains its form

$$\frac{\partial\tilde{\chi}}{\partial t} + J(\tilde{\chi};\psi) = 0; \quad \tilde{\chi}(\boldsymbol{r};0) = \tilde{\chi}_0(\boldsymbol{r}), \tag{35}$$

but the random fields χ; $\tilde{\chi}$ and θ; $\tilde{\theta}$ behave differently. The former have Gaussian (normal) stationary limit, whereas the latter are always log-normal. As above one could show the Gaussian equilibrium $\tilde{\chi}$ to be statistically equivalent to the random steady-state solution of (35)

$$\tilde{\chi}(\boldsymbol{r}) = -\lambda_1\psi(\boldsymbol{r})$$

with certain parameter λ_1. Hence follows the correlation-function for $\tilde{\chi}$

$$B_{\tilde{\chi}} = \lambda_1^2 B_\psi,$$

and the cross-correlators

$$B_{\tilde{\chi}h} = -\lambda_1 B_{\psi h}; \qquad B_{\tilde{\chi}\psi} = -\lambda_1 B_\psi.$$

Let us observe that the coherent (δ-type) equilibrium ensemble of $\{\psi\}$ yields a similar coherent state of $\tilde{\chi}$ and $\tilde{\theta}$ with the same length scale.

Appendix A. Correlator splitting and derivation of the basic equations

We use the dynamic equations (1) for each of three factors in the triple correlator of the stationary ensemble ψ

$$\frac{\partial}{\partial t}\langle\Delta\psi(\boldsymbol{r}_1;t)\,\Delta\psi(\boldsymbol{r}_2;t)\,\Delta\psi(\boldsymbol{r}_3;t)\rangle = \{1\} + \{2\} + \{3\}, \tag{A.1}$$

where symbol $\{1\}$ denotes the quantity

$$\{1\} = \langle J(\Delta\psi(\boldsymbol{r}_1;t) + h(\boldsymbol{r}_1);\psi(\boldsymbol{r}_1;t))\,\Delta\psi(\boldsymbol{r}_2;t)\,\Delta\psi(\boldsymbol{r}_3;t)\rangle, \tag{A.2}$$

while $\{2\}$ and $\{3\}$ correspond to cyclic permutations of variables $(\boldsymbol{r}_1;\boldsymbol{r}_2;\boldsymbol{r}_3)$ in (A.2). Next we split the quartic correlator in the r.h.s. (A.1) in the product of pair correlators using Gaussianity

$$\begin{aligned}\{1\} &= \langle J(\Delta\psi(\boldsymbol{r}_1;t) + h(\boldsymbol{r}_1);\psi(\boldsymbol{r}_1;t))\,\Delta\psi(\boldsymbol{r}_2;t)\,\Delta\psi(\boldsymbol{r}_3;t)\rangle \\ &= \left\langle\frac{\partial[\Delta\psi(\boldsymbol{r}_1;t)+h(\boldsymbol{r}_1)]}{\partial x_1}\,\Delta\psi(\boldsymbol{r}_2;t)\right\rangle\left\langle\frac{\partial\psi(\boldsymbol{r}_1;t)}{\partial y_1}\,\Delta\psi(\boldsymbol{r}_3;t)\right\rangle \\ &+ \left\langle\frac{\partial[\Delta\psi(\boldsymbol{r}_1;t)+h(\boldsymbol{r}_1)]}{\partial x_1}\,\Delta\psi(\boldsymbol{r}_3;t)\right\rangle\left\langle\frac{\partial\psi(\boldsymbol{r}_1;t)}{\partial y_1}\,\Delta\psi(\boldsymbol{r}_2;t)\right\rangle \\ &- \left\langle\frac{\partial[\Delta\psi(\boldsymbol{r}_1;t)+h(\boldsymbol{r}_1)]}{\partial y_1}\,\Delta\psi(\boldsymbol{r}_2;t)\right\rangle\left\langle\frac{\partial\psi(\boldsymbol{r}_1;t)}{\partial x_1}\,\Delta\psi(\boldsymbol{r}_3;t)\right\rangle \\ &- \left\langle\frac{\partial[\Delta\psi(\boldsymbol{r}_1;t)+h(\boldsymbol{r}_1)]}{\partial y_1}\,\Delta\psi(\boldsymbol{r}_3;t)\right\rangle\left\langle\frac{\partial\psi(\boldsymbol{r}_1;t)}{\partial x_1}\,\Delta\psi(\boldsymbol{r}_2;t)\right\rangle.\end{aligned} \tag{A.3}$$

Here we dropped the product terms containing $\langle J\,(\Delta\psi\,(\boldsymbol{r})+h;\,\psi\,(\boldsymbol{r}))\rangle$ as it vanishes after the ensemble averaging. Indeed, $J\,(\Delta\psi+h;\,\psi)$ represents the r.h.s. of the vorticity equation

$$\dot{\omega}=J\,(\omega;\,\psi)\,,$$

where $\omega=\Delta\psi+h$. But the latter is also written in terms of velocities

$$\frac{\partial}{\partial t}\omega+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{u}\omega)=0.$$

So $\langle J\,(\omega;\,\psi)\rangle=(\partial/\partial\boldsymbol{r})\cdot\langle\boldsymbol{u}\,\omega\rangle=0$ due to the statistical homogeneity of fields $\psi;\,\omega;\,\boldsymbol{u}$.

Using that same statistical homogeneity of ψ and h one can replace the r_1-derivatives $(\partial/\partial x_1;\,\partial/\partial y_1)$in (A.3) by the $\boldsymbol{r}_2$- and $\boldsymbol{r}_3$-derivatives. Then (A.3) takes the form

$$\begin{aligned}
\{1\}=&\frac{\partial}{\partial x_2}\,\langle[\Delta\psi\,(\boldsymbol{r}_1;t)+h\,(\boldsymbol{r}_1)]\;\;\Delta\psi\,(\boldsymbol{r}_2;t)\rangle\,\frac{\partial}{\partial y_3}\,\langle\psi\,(\boldsymbol{r}_1;t)\;\;\Delta\psi\,(\boldsymbol{r}_3;t)\rangle\\
&+\frac{\partial}{\partial x_3}\,\langle[\Delta\psi\,(\boldsymbol{r}_1;t)+h\,(\boldsymbol{r}_1)]\;\;\Delta\psi\,(\boldsymbol{r}_3;t)\rangle\,\frac{\partial}{\partial y_2}\,\langle\psi\,(\boldsymbol{r}_1;t)\;\;\Delta\psi\,(\boldsymbol{r}_2;t)\rangle\\
&-\frac{\partial}{\partial y_2}\,\langle[\Delta\psi\,(\boldsymbol{r}_1;t)+h\,(\boldsymbol{r}_1)]\;\;\Delta\psi\,(\boldsymbol{r}_2;t)\rangle\,\frac{\partial}{\partial x_3}\,\langle\psi\,(\boldsymbol{r}_1;t)\;\;\Delta\psi\,(\boldsymbol{r}_3;t)\rangle\\
&-\frac{\partial}{\partial y_3}\,\langle[\Delta\psi\,(\boldsymbol{r}_1;t)+h\,(\boldsymbol{r}_1)]\;\;\Delta\psi\,(\boldsymbol{r}_3;t)\rangle\,\frac{\partial}{\partial x_2}\,\langle\psi\,(\boldsymbol{r}_1;t)\;\;\Delta\psi\,(\boldsymbol{r}_2;t)\rangle\,.
\end{aligned}$$

The latter can be recast through the correlators and cross-correlators of fields ψ and h

$$\begin{aligned}
\{1\}=&\left(\frac{\partial^2}{\partial x_2\partial y_3}-\frac{\partial^2}{\partial y_2\partial x_3}\right)\Delta_2\left[\Delta_2 B_\psi\,(\boldsymbol{r}_1-\boldsymbol{r}_2)+B_{\psi h}\,(\boldsymbol{r}_1-\boldsymbol{r}_2)\right]\Delta_3 B_\psi\,(\boldsymbol{r}_1-\boldsymbol{r}_3)\\
&+\left(\frac{\partial^2}{\partial x_3\partial y_2}-\frac{\partial^2}{\partial y_3\partial x_2}\right)\Delta_3\left[\Delta_3 B_\psi\,(\boldsymbol{r}_1-\boldsymbol{r}_3)+B_{\psi h}\,(\boldsymbol{r}_1-\boldsymbol{r}_3)\right]\Delta_2 B_\psi\,(\boldsymbol{r}_1-\boldsymbol{r}_2)
\end{aligned}\tag{A.4}$$

Introducing vectors

$$\boldsymbol{q}_1=\boldsymbol{r}_1-\boldsymbol{r}_2;\qquad \boldsymbol{q}_2=\boldsymbol{r}_2-\boldsymbol{r}_3;\qquad \boldsymbol{q}_3=\boldsymbol{r}_3-\boldsymbol{r}_1.$$

and scalars $q_i=|\boldsymbol{q}_i|$ we replace all r-partials in (A.4) through the $\boldsymbol{q}$-partials

$$\frac{\partial}{\partial x_2}=-\frac{x_1-x_2}{q_1}\frac{\partial}{\partial q_1};\quad \frac{\partial}{\partial y_2}=-\frac{y_1-y_2}{q_1}\frac{\partial}{\partial q_1}\quad \frac{\partial}{\partial x_3}=\frac{x_3-x_1}{q_1}\frac{\partial}{\partial q_1};\quad \frac{\partial}{\partial y_3}=\frac{y_3-y_1}{q_1}\frac{\partial}{\partial q_1}.$$

The off-shot is the following equation

$$\begin{aligned}
\{1\}=&-(\boldsymbol{q}_1\wedge\boldsymbol{q}_3)\,\frac{1}{q_1q_3}\,\frac{\partial^2}{\partial q_1\partial q_3}\Delta_1\left\{\left[\Delta B_\psi\,(q_1)+B_{\psi h}\,(q_1)\right]\Delta_3 B_\psi\,(q_3)\right.\\
&\left.-\left[\Delta B_\psi\,(q_3)+B_{\psi h}\,(q_3)\right]B_\psi\,(q_1)\right\}\\
=&-(\boldsymbol{q}_3\wedge\boldsymbol{q}_1)\;X\,(q_3;q_1)\,.
\end{aligned}$$

Here we denote by $\boldsymbol{q}_1\wedge\boldsymbol{q}_3$ the wedge-product of two vectors (area of the parallelogram spanned by $\boldsymbol{q}_1;\,\boldsymbol{q}_3$). In the similar vein we treat two other equations for the {2} and {3}-terms of (A.1). The result is

$$(\boldsymbol{q}_3\wedge\boldsymbol{q}_1)\;X\,(q_3;q_1)+(\boldsymbol{q}_2\wedge\boldsymbol{q}_3)\;X\,(q_2;q_3)+(\boldsymbol{q}_1\wedge\boldsymbol{q}_2)\;X\,(q_1;q_2)=0.$$

Once the common factor

$$q_3 \wedge q_1 = q_2 \wedge q_3 = q_1 \wedge q_2$$

is dropped (remembering $q_1 + q_2 + q_3 = 0$) we end up with the basic functional equation (11) for $X(p; q)$ (10).

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References

[1] J. Pedlosky, Geophysical Fluid Dynamics (Springer, New York, 1982).
[2] R. Kraichnan, Inertial waves in two-dimensional turbulence, Phys. Fluids 10 (1967) 1417–1423.
[3] R. Kraichnan, Statistical dynamics of two-dimensional flows, J. Fluid Mech. 67 (1975) 155–175.
[4] R. Kraichnan and D. Montgomery, Two-dimensional turbulence, Rep. Prog. Phys. 43 (1980) 547–619.
[5] V. Klyatskin, On statistical theory of two-dimensional turbulence, J. Appl. Math. and Mech. 33 (5) (1969) 864–866.
[6] R. Salmon, G. Holloway and M. Hendershott, The equilibrium statistical mechanics of simple quasigeostrophic models, J. Fluid Mech. 75 (4) (1976) 691–703.
[7] G. Holloway, Eddies, waves, circulation and mixing: statistical geofluid mechanics, Ann. Rev. Fluid Mech. 18 (1986) 91–147.
[8] G. Carnevale and J. Frederiksen, Nonlinear stability and statistical mechanics of flow over topography, J. Fluid Mech. 175 (1987) 157–181.
[9] J. Miller, P. Weichman and M. Cross, Statistical mechanics, Euler's equation and Jupiter's red spot, Phys. Rev. A 45 (1992) 2328.
[10] J.R. Herring, On the statistical theory of two-dimensional topographic turbulence, J. Atmos. Sci. 34 (11) (1977) 1731–1750.
[11] J.R. Herring, Stochastic modeling of turbulent flows, in: Stochastic Modeling in Physical Oceanography, eds. Adler et al. (Birkhauser, Basel, 1996).
[12] A. Chorin, Vorticity and turbulence (Springer, New York, 1994).
[13] M.B. Isichenko, Percolation, statistical topography, and transport in random media, Rev. Mod. Phys. 64 (4) (1992) 961-1043.
[14] V. Zakharov, G. Falkovich and V. L'vov, Kolmogorov Spectra of Wave-turbulence (Springer, Berlin, 1992).
[15] A. Monin and A. Yaglom, Statistical fluid mechanics, (MIT Press, Cambridge MA, 1980).
[16] V. Klyatskin, Statistical description of dynamical systems with fluctuating parameters (Nauka, Moscow, 1975) (in Russian).
[17] V. Klyatskin, Ondes et equations stochastique dans les milieux aleatorement non homogenes, Edition du Physique, Besançon-Cedex, 1985).
[18] D. Gurarie, Symmetries and conservation laws of two-dimensional hydrodynamics, Physica D 86 (1995) 621–636.
[19] E. Hopf, Statistical hydrodynamics and functional calculus, J. Ration. Mech. Anal. 1 (1952) 87.
[20] N.P. Fofonoff, Steady flows in a frictionless homogeneous ocean, J. Mar. Res. 13 (1954) 254.