

Semiclassical eigenvalues and shape problems on surfaces of revolution

David Gurarie

Department of Mathematics, CWRU, Cleveland, Ohio 44106

(Received 18 April 1994; accepted for publication 6 October 1994)

Mark Kac [“Can one hear the shape of drums?” *Am. Math. Monthly* **73**, 1–23 (1966)] asked if the shape of a region $\Omega \subset \mathbb{R}^n$ could be determined from its sound (spectrum of the Laplacian Δ_Ω). He proved the conjecture for special classes of domains, including polygons and balls in \mathbb{R}^n . Similar problems could be raised in other geometric contexts, including “shape of metric” for Laplacians on manifolds or “shape of potential” for Schrödinger operators $H_V = \Delta + V$. The latter two problems for surfaces of revolution Σ are addressed. An explicit reconstruction procedure will be outlined that leads from the joint spectrum of $H = \Delta$ or $\Delta + V$ and the angular momentum algebra $\mathfrak{so}(n)$ to the “shape of Σ ” and “ V ,” respectively. The metric result applies to generic surfaces of revolution, while the Schrödinger result allows all zonal (axisymmetric) potentials V on Σ . So the work extends the Kac’s “ n -ball” result as well as the n -sphere “zonal Schrödinger theory” [D. Gurarie, (a) “Averaging methods in spectral theory of Schrödinger operators,” *Maximal Principles and Eigenvalues Problems in Differential Equations*, Pitman Research Notes Vol. 175, edited by P. W. Schaefer (Pitman, New York, 1980), pp. 167–77; (b) “Inverse spectral problem for the two-sphere Schrödinger operators with zonal potential,” *Lett. Math. Phys.* **16**, 313–323 (1990); (c) “Zonal Schrödinger operators on the n -sphere: Inverse spectral problem and rigidity,” *Commun. Math. Phys.* **131**, 571–603 (1990)]. © 1995 American Institute of Physics.

I. INTRODUCTION

An n -dimensional hypersurface of revolution Σ in space $\mathbb{R}^{n+1} = \{(\mathbf{z}; t) : \mathbf{z} \in \mathbb{R}^n; t \in \mathbb{R}\}$ is obtained by rotating a plane curve $z_1 = R(t)$ about the t axis, the defining equation being $|\mathbf{z}| = R(t)$ (see Fig. 1). In what follows it will be convenient to parametrize the curve by its arclength: $dx^2 = dt^2 + dR^2$, and write radius as function of x $R = R(x)$. The t coordinate can then be computed from the differential $dt = \sqrt{1 - (dR/dx)^2} dx$. We shall consider surfaces that are closed (spanned by closed curves) or bounded by hyperplanes $A \leq t \leq B$ with suitable rotationally symmetric boundary conditions. Surface Σ has a natural Riemannian metric and the corresponding Laplacian. Furthermore, it possesses a symmetry group $\mathbf{SO}(n)$ of rotations about the t axis.

We shall study asymptotics of large eigenvalues of the Laplacian Δ , and those of the Schrödinger operator $H_V = \Delta + V(z)$, with an axisymmetric potential V . The main result of the article is to show that properly selected spectral data of operators Δ and H_V determine the classical action-function $S(E)$ and higher-order Bohr–Sommerfeld corrections of certain reduced 1-D Hamiltonians: $h(x; p) = p^2 + q(x) = E$, at all energy levels. Here potential $q = 1/R^2$ depends on the metric coefficient $R(t)$ for the Laplacian problem, while the Schrödinger problem involves V and a certain (nonlinear) transform of R . As a consequence we are able to recover the metric and the potential on the surface of revolution from the joint spectrum of $H = \Delta$ or H_V and the angular momentum algebra $\mathcal{H} = \mathfrak{so}(n)$. Let us remark that the knowledge of the action function S_q alone is sufficient to recover special classes of potentials, like monotone functions $\{q\}$, respectively, monotone expanding/contracting R “horns” (this result was obtained in an earlier Preprint³). The multiwell potential case, however, is more involved. It requires the higher-order Bohr–Sommerfeld corrections,^{4,5} the proper choice of “spectral clusters,” and certain screening procedures similar to the Schrödinger theory.²

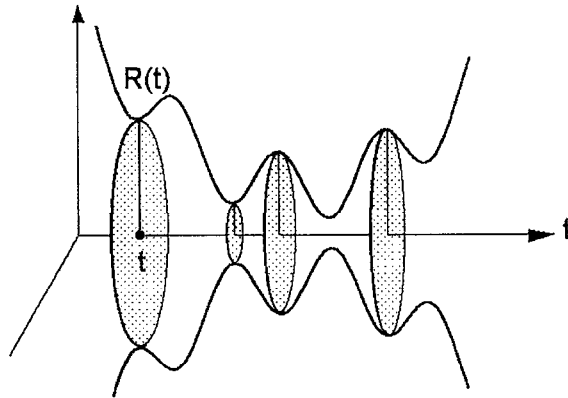


FIG. 1. A typical surface of revolution defined by function $R(t)$.

Our work extends the original Kac result for the disk and the solid sphere to general surfaces of revolution, but the methods are different. Kac exploited the classical heat-Weyl spectral invariants $\{b_0; b_1\}$, obtained from the asymptotic expansion of the heat kernel, $\text{tr}(e^{-t\Delta}) \sim t^{-n/2} \{b_0 + t^{1/2} b_1 + \dots\}$. The first term b_0 is proportional to $\text{vol}(\Sigma)$, while b_1 is proportional to the boundary area of $\partial\Sigma$. The isoperimetric inequality between $\text{vol}(\Sigma)$ and $\text{area}(\partial\Sigma)$ would then imply uniqueness (rigidity) of the solid sphere. We exploit the reduction of the multi-D Laplacian/Schrödinger operator to a sequence of 1-D semiclassical problems, where the quantization rules like Wentzel–Kramers–Brillouin (WKB) and Bohr–Sommerfeld apply. Our emphasis is also somewhat different, it is less on Kac’s original formulation “*Can one hear the shape ... from spectrum,*” but rather on “*How to do it.*” In other words, we explicitly reconstruct surface Σ (i.e., metric coefficient R) from the joint spectrum. After the inverse metric problem is solved we turn to the Schrödinger problem, and show how to determine potential V from $\text{spec}(V)$ for a given surface Σ . The latter result extends our earlier work on the inverse Schrödinger problem on the sphere.²

Let us remark that our results rely heavily on the joint $(H; \mathcal{M})$ spectrum. One might wonder whether $\text{spec}(H)$ alone would suffice to recover Σ and V . That is likely to be the case for higher-D surfaces, $\dim(\Sigma) \geq 3$, due to different spectral multiplicities of eigenvalues $\{\lambda\}$ that belong to different symmetry sectors, irreducible components of group $\mathbf{SO}(n)$ on the cross-sectional sphere S^{n-1} . Indeed, multiplicities $\{d_m = \text{dimension of spherical harmonics of degree } m\}$ increase with m in dimensions $n - 1 \geq 2$, but remain constant for 2-D surfaces (1-D sphere). Different multiplicities would allow one to disentangle different spectral branches (sectors) of $\text{spec}(H)$, if one knew that these branches did not overlap, in other words all or “most” of the eigenspaces of operator H were irreducible under the $\mathbf{SO}(n)$ action. The absence of accidental degeneracies however is still largely unresolved problem. So the complete solution of the inverse problem hinges upon the resolution of spectral degeneracies.

We hope to address these issues elsewhere.

II. THE MAIN RESULTS

A typical surface of revolution Σ is obtained by rotating curve $R = R(t)$ or $R(x)$ by the group $\mathbf{SO}(n)$ about the t axis

$$u:(z;t) \rightarrow (z^u;t); u \in \mathbf{SO}(n).$$

We shall parametrize surface Σ by two variables: the arclength parameter x along the generating curve $R(x)$ and the set of spherical variables θ on the cross-sectional sphere

$$\{(z;x):|z|=R(x)\} \approx S^{n-1}.$$

The corresponding Riemannian metric is $dx^2 + R^2 d\theta^2$ and the Laplacian on Σ takes the form

$$\Delta = \frac{1}{R^{n-1}} \partial_x R^{n-1} \partial_x + \frac{1}{R^2} \Delta_S,$$

where Δ_S is the spherical Laplacian on S^{n-1} . Operator Δ acts on the Hilbert space $\mathcal{L} = L^2(\Sigma; dV)$ with the volume element $dV = R^{n-1} dx dS(\theta)$. So conjugation with the multiplication operator $R^{(n-1)/2}$ brings Δ into a symmetric (Sturm–Liouville) form

$$H = \partial_x^2 - W(x) + q \Delta_S \tag{1}$$

in space $L^2(dx; d\theta)$ where one has the potentials

$$q = \frac{1}{R^2} \quad \text{and} \quad W(x) = \frac{(n-1)(n-3)}{4} \left(\frac{R'}{R}\right)^2 - \frac{n-1}{2} \left(\frac{R'}{R}\right). \tag{2}$$

We are interested in the joint spectrum of Δ and the angular momentum algebra $\mathcal{M} = \mathfrak{so}(n)$ in $L^2(\Sigma)$. Such a spectrum is naturally labeled by two parameters $\{(m; \lambda_{mk})\}$, where m gives the angular momentum value on a suitable subspace $\mathcal{L}_m \subset L^2(\Sigma)$, while sequence $\{\lambda_{mk}\}_k$ (for each fixed m) represents the spectrum on the reduced Laplacian $H_m = \Delta|_{\mathcal{L}_m}$. We recall that space $L^2(\Sigma)$ is decomposed into the direct sum of isotropic $\mathbf{SO}(n)$ -components

$$\mathcal{L} = \bigoplus_m \mathcal{L}_m. \tag{3}$$

For each $\mathcal{L}_m \approx \mathcal{H}_m \otimes L^2(dx)$ —spherical harmonics of degree m on S^{n-1} tensored by L^2 functions in the lateral variable. Decomposition (3) allows to separate variables in Eq. (1) and to write eigenfunctions in the product form

$$\Psi(x; \theta) = \psi(x) Y_m(\theta); \quad Y_m \in \mathcal{H}_m.$$

Hence operator Δ is reduced to a family of 1-D Sturm–Liouville (SL) operators in variable x

$$H_m = -\partial_x^2 + m(m+n-2)q + W(x). \tag{4}$$

The joint $(\Delta; \mathcal{M})$ spectrum can thus be viewed as a subset of the energy-momentum (spectral) plane made of points $\{(\lambda; m): m=0; \pm 1; \pm 2; \dots\}$, where each horizontal line (level m) contains spectrum $\{\lambda_{mk}\}_k$ of the reduced SL operator H_m (4) (see Fig. 2). As is customary in the SL theory we shall rescale eigenvalues from $\{\lambda\}$ to $\{\sqrt{\lambda}\}$, and view them in the $(\sqrt{\lambda}; m)$ plane. Our main result is

Theorem 1: *A generic surface of revolution $\Sigma = \Sigma_R$, i.e., generic radius function $R(x)$, can be uniquely and explicitly reconstructed from the joint spectrum $\{(\sqrt{\lambda_{mk}}; m)\}$ of the Laplacian and the angular momentum algebra $\mathcal{M} = \mathfrak{so}(n)$.*

The meaning of “generic” will be explained later on, it involves certain properties of the action functional S_q of the corresponding classical Hamiltonian. Suffices it to say that Theorem 1 applies to a broad classes of functions: polynomials, real analytic functions $\{R\}$, etc. After the main result is established for Laplacians we turn to Schrödinger operators $H_V = -\Delta + V$ on surfaces of revolution with axisymmetric (zonal) potentials $\{V = V(x)\}$. Such operators H also pos-

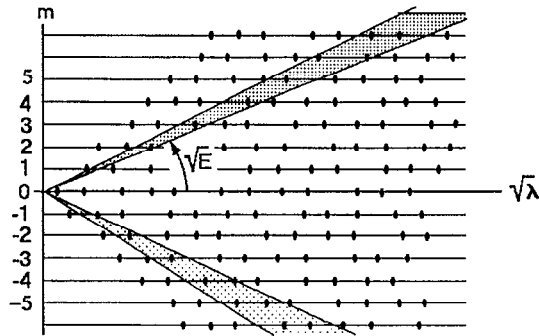


FIG. 2. The joint $(\Delta; M)$ spectrum in the energy-momentum plane and the narrow-angle clustering.

sess rotational symmetry, $\mathfrak{so}(n)$, and allow reduction to a sequence of 1-D problems $\{H_m\}$. The input data for the inverse problem in the Schrödinger case is made of differences, and spectral shifts $\{\mu_{mk} = \lambda_{mk}(H) - \lambda_{mk}(\Delta)\}$ between the Schrödinger eigenvalues and those of the Laplacian. As above we arrange $\{\mu_{mk}\}$ into spectral branches according to the angular momentum.

Theorem 2: *Spectral shifts $\{(m; \mu_{mk})\}$ of the Schrödinger operator H_V labeled by the angular momentum algebra $M = \mathfrak{so}(n)$ determine uniquely and explicitly a generic axisymmetric (zonal) potential $V(x)$ on Σ .*

We shall first outline the metric result.

III. METRIC PROBLEM

A. Semiclassical analysis

The eigenvalue problem $H_m[\psi] = \lambda\psi$ for operator (4) with large m and λ can be recast into a semiclassical form dividing it by coupling constant $(m + (n-2)/2)^2 = 1/\hbar^2$ in front of q . For the sake of notation we shall relabel $(m + (n-2)/2)^2$ as m^2 so the Planck $\hbar = 1/m$, and the resulting semiclassical problem becomes

$$(-\hbar^2 \partial^2 + q + \hbar^2 W)[\psi] = E\psi.$$

Here $q = 1/R^2$ and the rescaled energy $E = \lambda/m^2$. In what follows we shall drop the higher-order potential perturbation $\hbar^2 W$ and concentrate on $-\hbar^2 \partial^2 + q = E$. Our method involves several steps:

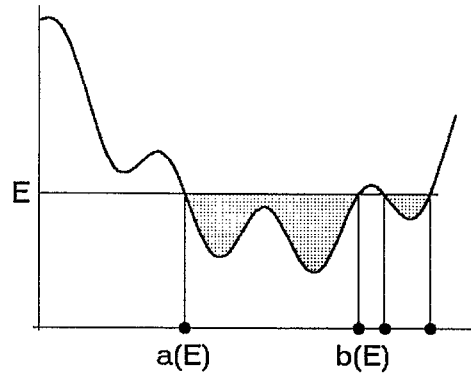
(1) A suitable choice of narrow-angle spectral clusters $\{\Lambda_m = \Lambda_m(E)\}$ in a given energy-direction E ;

(2) Semiclassical analysis of eigenvalues $\{\sqrt{\lambda}\}$ in Λ_m based on the Bohr–Sommerfeld quantization with higher-order corrections. Cluster Λ_m will be shown to consist of finitely many distorted arithmetic sequences $\{\Lambda_{m,j}; j = 1; 2; \dots\}$ each one corresponding to a single potential well of q at level E .

(3) A proper screening procedure² will be used to disentangle sequences $\{\Lambda_{m,j}\}$ to read off the requisite geometric data, namely, the right and left slope functions $\{a'(E); b'(E)\}$ of each potential well (Fig. 3). Hence we recover potential q and the radius function $R = 1/\sqrt{q}$.

We shall elaborate the steps. The narrow angle clusters $\{\Lambda_m\}$ are chosen along a given asymptotic direction $\sqrt{E} = \sqrt{\lambda}/m$ (Fig. 2). Their size

$$|\Lambda_m| = \#\{k: |\sqrt{\lambda_{mk}} - m\sqrt{E}| \leq Cm^\nu\} \quad (5)$$

FIG. 3. Potential wells of the q cutoff by the energy level E .

is controlled by two parameters C, ν to be chosen later. The cluster size will also increase as $\mathcal{O}(m^\nu)$ as $m \rightarrow \infty$. The generalized Bohr–Sommerfeld quantization formula to be used^{4,5} has two terms

$$S(E) + \hbar^2 T(E) + \dots = \hbar \pi (k + \frac{1}{2}). \quad (6)$$

Here $\hbar = 1/m$, energy $E = \lambda/m^2$ while the left hand side involves the classical action S and its first correction T

$$S(E) = \int \sqrt{E - q} \, dx, \quad T(E) = \frac{1}{96} \int \frac{q''}{(E - q)^{3/2}} \, dx. \quad (7)$$

The T -integral (7) is defined via analytic regularization of $\int q''(E - q)^{-s}$ in complex parameter s . Semiclassical formulas (6) are valid for each potential well of q below a given (fixed) energy level $q = E$. In other words formula (6) defines a sequence of eigenvalues “localized” to the j th well of q , integration (7) extending from the left to the right wall of the well (see Fig. 3)

$$a = a_j(E) \leq x \leq b = b_j(E).$$

Different potential wells will contribute different sequences of eigenvalues at any given level E , so one needs to disentangle contributions of individual wells to clusters $\{\Lambda_m\}$ to get a geometric data describing each one of them. All semiclassical data will be expressed through two special combinations of the right and left slope functions $\{a'(E); b'(E)\}$ of the well (Fig. 3). Namely, we introduce auxiliary functions

$$\Phi(E) = b' - a', \quad \Psi(E) = \frac{1}{b'} - \frac{1}{a'}. \quad (8)$$

The two action terms (7) can be shown to represent fractional derivatives of $\Phi; \Psi$

$$S = \Gamma(-\frac{3}{2}) \Phi^{(-3/2)}, \quad T = \Gamma(-\frac{1}{2}) \Psi^{(3/2)}.$$

[We recall the definition of fractional derivative of order $\alpha: f(x) \rightarrow f^{(\alpha)}(x) = [1/\Gamma(-\alpha)] \int_0^x (x - \xi)^{-\alpha-1} f(\xi) d\xi$.] Furthermore, both action functions S and T could be expanded in fractional powers of small increment δE

$$\begin{aligned}\delta S &= S_1 \delta E + S_2 (\delta E)^{3/2} + S_3 (\delta E)^2 + \dots, \\ \delta T &= T_1 \delta E + T_2 (\delta E)^{3/2} + \dots,\end{aligned}\tag{9}$$

with certain coefficients $\{S_1; S_2; \dots\}$ and $\{T_1; T_2; \dots\}$ depending on E that could be computed explicitly through Φ and Ψ . We shall list the first few of them

$$\begin{aligned}S_1(E) &= \frac{dS}{dE} = \int_a^b \frac{dx}{\sqrt{E-q}} = \text{Const } \Phi^{(-1/2)}, \\ T_1(E) &= \frac{dT}{dE} = -\frac{1}{46} \int_a^b \frac{q'' dx}{(E-q)^{5/2}} = \text{Const } \Psi^{(5/2)}.\end{aligned}\tag{10}$$

Let us remark that S_1 is the classical period of oscillation of a particle in the potential well at a fixed energy. All other coefficients of the S series are also fractional (integral or half integral) derivatives of Φ , e.g.,

$$S_2 = \frac{2}{3}(b' - a') = \frac{2}{3}\Phi, \quad \text{etc.}$$

Similarly the T series is made of fractional derivatives of Ψ . Clearly, the knowledge of Φ and Ψ (8) is sufficient to recover the left and right slopes $\{a'(E); b'(E)\}$ of a given potential well of q .

We shall apply the semiclassical asymptotics (6) of cluster eigenvalues (5) to identify all potential wells of q at a given level E , and then to read off the coefficients $\{S_1(E); T_1(E)\}$ of Eq. (10) for each well. As a consequence we shall reconstruct functions $\Phi; \Psi$, hence the left/right slopes $a'; b'$ of each well. The key to our analysis is

Proposition 3:

(1) Clusters $\{\Lambda_m(E)\}$ of Eq. (5) are made of a finite number of distorted arithmetic sequences $\{\Lambda_{m,j}; j=1,2,\dots\}$ each one associated to a j th potential well of H_q at level E .

(2) Eigenvalues $\sqrt{\lambda_{mk}}$ in the j th subcluster $\Lambda_{m,j}$ (the j th well) admit the following asymptotics at large $m; k$

$$\frac{\sqrt{\lambda_{mk}}}{m} - \sqrt{E} = F\left(\pi \frac{k}{m}\right) + \frac{1}{m^2} G\left(\pi \frac{k}{m}\right) + O\left(\frac{1}{m^4}\right),\tag{11}$$

where functions F and G are expressed through the two action terms $S; T$ of Eq. (7) as

$$F = S^{-1}; \quad G = S^{-1} \circ T \circ S^{-1}.$$

(3) As a consequence of Eq. (2) functions F and G can be expanded in fractional powers of variable E , with coefficients that involve the S and the T series (9)–(10).

We shall write the first few of them at point E with small increment $x = \delta E$

$$\begin{aligned}F(E+x) &= \frac{1}{S_1} x - \frac{S_2}{S_1^{3/2}} x^{3/2} + \frac{3S_2 - S_1 S_3}{2S_1^4} x^2 + \dots, \\ G(E+x) &= -\frac{T_1}{S_1^2} x + \frac{5}{2} \frac{S_2 T_1}{S_1^{7/2}} x^{3/2} + \dots.\end{aligned}$$

The proof of the proposition exploits the two-term Bohr–Sommerfeld quantization formula (7) and the series expansion (9).

The proposition exhibits the requisite geometric data (slopes $a'; b'$) encoded in the principal F -coefficient $F_1 = 1/S_{1,j}$ and the principal G -coefficient $G_1 = T_1/S_1^2$. However, to pull out such data from asymptotics (11) we need to disentangle constituent subsequences $\{\Lambda_{m,j}; j = 1, 2, \dots\}$ in the m th cluster by a suitable screening procedure.

B. Screening and reconstruction of slopes

We pick a generic potential q from a suitable class (polynomial, analytic, etc.) and observe that for generic energy-level $q = E$ potential wells of q

$$\{a_j \leq x \leq b_j\} (1 \leq j \leq N)$$

should have rationally independent periods $\{S_{1,j}\}$ of Eq. (10). So cluster Λ_m is made of the distorted arithmetic sequences

$$\Lambda_{m,j} = \left\{ \sqrt{\lambda_{m,n}} \approx D_j \frac{n}{m} + \dots \right\},$$

with rationally independent differences $\{D_j = 1/S_{1,j}\}$. The situation closely resembles the n -sphere Schrödinger case.² There we developed an argument (based on rational independence) that allows to disentangle such unions and produce numbers $\{D_j\}$. Since coefficient $S_1 = \Phi^{(-1/2)}$, we at once recover function $\Phi = b' - a'$ of Eq. (8), as well as the higher-order coefficients of F and S (9). Once a sufficient number of the S terms is computed we can proceed to the principal T coefficient $T_1(E)$. The latter appears as $1/m^2$ -order corrections in Eq. (11). So one only need to subtract the proper number of intermediate "fractional powers" of $1/m$ in the F series to get to the order $1/m^2$. A simple order counting near cluster edges $\{\sqrt{\lambda_{m,k}}; k = O(m^{1-\nu})\}$ gives us both a proper choice of the parameter ν of Eq. (5), $\frac{4}{3} < \nu < 1$, and the number of intermediate terms of the F series

$$F_5(x) = F_1 x + F_2 x^{3/2} + F_3 x^2 + F_4 x^{5/2} + F_5 x^3.$$

In other words the difference $\sqrt{\lambda_{m,n}} - F_5(n/m)$ will have the principal-order part $G_1 n/m^3$. Hence we immediately read off $G_1(E) = T_1/S_1^2$, the coefficient $T_1 = G_1 S_1^2$, and consequently function Ψ of Eq. (8)

$$\Psi(E) = \frac{1}{b'} - \frac{1}{a'} = T_1^{(5/2)}.$$

Once Φ and Ψ are found we obtain the right and left walls of the j th well that determine the shape of potential q , hence the surface of revolution. Namely,

$$\begin{aligned} a_j(E) &= \int \left\{ -\Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE, \\ b_j(E) &= \int \left\{ \Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE \end{aligned} \tag{12}$$

for the j th well. The last step is to patch different potential wells produced in Eq. (12) into a single function $q(x)$.

C. Patching and gluing

Let us remark that the "wall functions" $\{a; b\}$ are determined by Eq. (12) only up to constants. To fix the constants and glue wells together we proceed as follows.

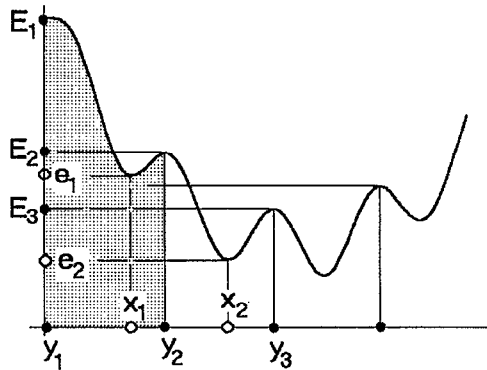


FIG. 4. Monotone branches (walls) between adjacent critical points.

(1) Locate minimal values $\{e_j = q(x_j)\}$, where $\{x_j\}$ are local minima (bottom of the well). Those are characterized by the change from the two-sided “almost arithmetic sequences” (11) to the one-sided oscillator spectra

$$\{\sqrt{\lambda_{m,k}} = \sqrt{e_j + \omega_j(2k+1)} : k=0; 1 \dots\},$$

with frequencies $\omega_j = q''(x_j)$. Similarly we can determine local maxima $\{E_j = q(y_j)\}$. Once again different one-sided sequences appear above and below such energy values \sqrt{E} (see Fig. 4). Let us remark that generic (Morse) functions $\{q\}$ have all critical points $\{x_j; y_j\}$ isolated and nonsingular ($q'' \neq 0$) and critical values $\{e_j; E_j\}$ distinct.

(2) After the critical values $\{e_j; E_j\}$ are determined it remains to fix the critical points $\{x_j; y_j\}$. Those could be fixed by obvious conditions on the monotone branches (walls) of q connecting the nearest extrema

$$\begin{aligned} a_j(E) &= x_j + \frac{1}{2} \int_{e_j}^E \left\{ -\Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE, \\ b_j(E) &= x_j + \frac{1}{2} \int_{e_j}^E \left\{ \Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE. \end{aligned} \quad (13)$$

The branches will rigidly fix the precise distances between adjacent critical points $\{y_j - x_j\}$ and $\{x_{j+1} - y_j\}$ by Eq. (13) after a possibly nonunique discrete (combinatorial) step of distributing the critical values $\{e_j; E_j\}$ in the increasing order of critical points $\{\dots < x_j < y_j < x_{j+1} < \dots\}$. Namely,

$$\begin{aligned} y_j - x_j &= \frac{1}{2} \int_{e_j}^{E_j} \left\{ \Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE, \\ x_{j+1} - y_j &= \frac{1}{2} \int_{E_j}^{e_{j+1}} \left\{ -\Phi + \sqrt{\Phi^2 - 4 \frac{\Phi}{\Psi}} \right\} dE \end{aligned}$$

will rigidly fix $\{x_j; y_j\}$. The former integration is over the right b -wall of the j th well, while the latter goes along the left a -wall of the adjacent $(j+1)$ th well (see Fig. 4).

This completes the proof of the metric problem.

IV. INVERSE SCHRÖDINGER PROBLEM

After potential q and the corresponding metric coefficient R of surface Σ are determined in the previous section we can turn to the inverse Schrödinger problem with zonal (axisymmetric) potential V on Σ . Perturbing the Laplacian Δ_Σ with V produces a sequence of spectral shifts $\lambda_{mk} \rightarrow \lambda_{mk} + \mu_{mk}$, for each spherically reduced perturbed operator

$$H_m + V = (\partial^2 + m^2 q) + V.$$

Our goal is to compute the asymptotics of spectral shifts $\{\mu_{mk}\}$ in terms of potential V and the known spectrum of H_m . The procedure exploits a modification of the averaging method developed in the n -sphere theory.^{6-8,2}

In the n -sphere case it was convenient to replace the Laplacian and the Schrödinger operator by their square roots, i.e.,

$$A = \sqrt{-\Delta + ((n-1)/2)^2} - \left(\frac{n-1}{2}\right) \quad \text{and} \quad B = \sqrt{-\Delta + V} - A.$$

The latter can be expanded in terms of decreasing orders (see Ref. 2c)

$$B = \frac{1}{2}A^{-1}V - \frac{1}{4}A^{-2} \text{ad}_A[V] + \frac{1}{8}(A^{-3} \text{ad}_A^2[V] - A^{-2}VA^{-1}V) + \dots, \quad (14)$$

where ad_A denotes the adjoint action (commutator) of A on the algebra of operators $\text{ad}_A(X) = AX - XA$. The main advantage of passing from $\Delta + V$ to $A + B = \sqrt{\Delta + V}$ in the sphere case is the periodicity of A . Namely, $\text{spec } A = \{0; 1; 2 \dots\}$, equivalently the unitary group $\{\exp(itH)\}$ is 2π periodic. The same holds for the classical geodesic flow on S^n generated by the Hamiltonian function $\sigma_A(x; p) = \sqrt{\Sigma g^{ij}p_i p_j}$ —symbol of operator A , where $\{g^{ij}\}$ are coefficients of the metric tensor (kinetic energy form) on the cotangent phase-space $T^*(S^n)$.

In our case the role of the square-root function will be played by the classical action-function $S(E) = S_q(E)$ of Eq. (7). In other words we define the operator

$$A_m = mS\left(\frac{1}{m^2} H_m\right)$$

and write the original Hamiltonian as

$$H_m + V = m^2 S^{-1}\left(\frac{1}{m} A_m + B\right),$$

with B expanded in a manner similar to Eq. (14). The role of the Taylor coefficients $\{\frac{1}{2}; -\frac{1}{4}; \frac{1}{8}; \dots\}$ of the square-root function will be played now by the Taylor coefficients of the action-function $S(E)$ at a fixed energy value E . As in the spherical case both operator A and the corresponding classical Hamiltonian flow generated by $\sigma_A = S(p^2 + q(x))$ are 2π periodic. Once the problem is recast the “ $A + B$ ” form the averaging machinery^{6-8,2} cranks in. The idea is to replace perturbation B by its average

$$\bar{B} = \frac{1}{2\pi} \int_0^{2\pi} e^{itA} B e^{-itA} dt.$$

Two operators $A + B$ and $A + \bar{B}$ are almost unitarily equivalent. Hence one gets an asymptotic expansion of spectral shifts $\mu_{km}(B) \approx \mu_{km}(\bar{B})$ in the form

$$\mu_{km} \approx \alpha \left(\frac{m}{k} \right) + \frac{1}{k} \beta \left(\frac{m}{k} \right) + \frac{1}{k^2} \gamma \left(\frac{m}{k} \right) + \dots, \quad \text{as } k \rightarrow \infty. \quad (15)$$

The coefficients $\{\alpha; \beta; \gamma; \dots\}$ represent certain (linear, bilinear, etc.) transforms of V . We shall compute the first one

$$\alpha(E) = \sigma_{\tilde{B}} = \frac{1}{2\pi} \int_0^{2\pi} V \circ \exp(t\tilde{\Xi}) dt = \frac{1}{T(E)} \int \frac{V(x) dx}{\sqrt{E-q}}. \quad (16)$$

Here $\exp(t\tilde{\Xi})$ is the Hamiltonian flow of symbol σ_A and $T(E)$ denotes the period of original Hamiltonian function $p^2 + V(x)$ at the energy level E . We shall call the transform defined by (16) the \mathcal{R} transform of V by analogy with the reduced Radon transform on the sphere, so

$$\mathcal{R}: V(x) \rightarrow \tilde{V}(E) = \frac{1}{T(E)} \int \frac{V(x) dx}{\sqrt{E-q}}. \quad (17)$$

Then the leading term of the asymptotic expansion (15) yields the values of \tilde{V} at all rationals $\{m/k\}$. Indeed

$$\mu_{km} \approx \tilde{V} \circ S^{-1} \left(\frac{m}{k} \right) + \mathcal{O} \left(\frac{1}{k^2} \right).$$

The latter holds uniformly for all $|m| < k$ (see Ref. 2). This shows how the asymptotic expansion of spectral shifts $\{\mu_{km}\}$ yields the \mathcal{R} transform of V composed with function S^{-1} . Both R and S are completely determined by the basic potential function $q(x)$. Hence the perturbation potential V is uniquely reconstructed from \tilde{V} by inverting the \mathcal{R} transform (17). We have thus proven Theorem 2.

We shall conclude the discussion with two comments.

(1) The transform (17) considered on the sphere becomes the standard reduced Radon transform of Ref. 2. In fact Theorem 2 extends the inverse spectral results of Ref. 2 to arbitrary surfaces of revolution. As in the former case (n -sphere) $\text{spec } H_V$ exhibits a multiscale structure, whose principal part has asymptotics (11) depending on the “metric potential” q , while the higher-order corrections (shifts) $\{\mu_{km}\}$ are given by the modified Radon transform \mathcal{R} and other related transforms of V .

(2) A recent article⁹ studied the inverse potential problem for radial functions $\{V(|x|)\}$ on Euclidean balls $\{|x| \leq 1\}$ in R^n . Here the problem is reduced to perturbations of the associated Legendre operators

$$H_m = \partial_r^2 + \frac{n-1}{r} \partial_r - \frac{m(m+n-1)}{r^2} + V.$$

The authors showed that for suitable pairs of the reduced operators $\{H_m; H_k\}$, depending on the parity of $\{k; m\}$ the joint isospectral class

$$\text{iso}(H_m + V) \cap \text{iso}(H_k + V),$$

i.e., the set of all $\{V\}$ that preserve both $\text{spec}(H_m + V)$ and $\text{spec}(H_k + V)$, is finite dimensional. Our result implies that the intersection of all isospectral classes

$$\bigcap_m \text{iso}(H_m + V)$$

is in fact unique on arbitrary surfaces of revolution.

Added in the proof: After the article was submitted P. Gilkey has informed us about an earlier work by Brüning and Heintze on the inverse metric problem on surfaces of revolution [Math. Ann. **269** (1984)]. They proved that a 2-D surface with an additional reflectional symmetry [even metric-coefficient $R(x)$ or $q(x)$ on $-1 < x < 1$] could be recovered from an invariant spectrum of the Laplacian, i.e., the $m=0$ branch of $\text{spec}(\Delta_{\Sigma})$. The argument was based on the reduction to a double (Dirichlet–Neumann) spectral problem for operator H_q on $[0;1]$, solved by the Gelfand–Levitan–Marchenko method to show uniqueness. Though both articles deal with similar problems the techniques and scope of the results of Brüning and Heintze are very different from ours.

- ¹M. Kac, “Can one hear the shape of the drum?,” Am. Math. Monthly **73**, 1–23 (1966).
- ²D. Gurarie, (a) “Averaging methods in spectral theory of Schrödinger operators,” *Maximal Principles and Eigenvalues Problems in Differential Equations*, Pitman Research Notes, Vol. 175, edited by P. W. Schaefer (Pitman, New York, 1988), pp. 167–177; (b) “Inverse spectral Problem for the two-sphere Schrödinger operators with zonal potentials,” Lett. Math. Phys. **16**, 313–323 (1988); (c) “Zonal Schrödinger operators on the n -sphere: Inverse spectral problem and rigidity,” Commun. Math. Phys. **131**, 571–603 (1990).
- ³D. Gurarie, “Semiclassical eigenvalues and inverse spectral problem on surfaces of revolution,” Preprint, 1993.
- ⁴N. Fröman, “Outline of a general theory for higher-order approximations of the JWKB type,” Ark. Fis. **32**, (1966); “Connection formulas for certain higher-order phase-integral approximations,” Ann. Phys. **61** (1970).
- ⁵S. Yu. Slavianov, *Asymptotics of Solutions of One-Dimensional Schrödinger Equations* (Leningrad University, Leningrad, Russia, 1990).
- ⁶A. Weinstein, “Asymptotics of eigenvalue clusters for the Laplacian plus a potential,” Duke Math. J. **44**, 883–892 (1977).
- ⁷V. Guillemin, (a) “Spectral theory on S^2 : Some open questions,” Adv. Math. **42**, 283–290 (1981); (b) “Some spectral results for the Laplacian on the n -sphere,” Adv. Math. **27**, 273–286 (1978).
- ⁸A. Uribe, (a) “A symbol calculus for a class of pseudodifferential operators on S^n ,” J. Funct. Anal. **59**, 535–556 (1984); (b) “Band invariants and closed trajectories on n -sphere,” Adv. Math. **58**, 3 (1985).
- ⁹R. Carlson and C. Shubin-Christ, “Spectral rigidity for radial Schrödinger operators,” Preprint, 1992.