5-7-91
56479

The Inverse Spectral Problem

DAVID GURARIE
Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106

In this paper we shall consider the eigenvalue problem

$$H[\psi] = \lambda\psi \tag{0.1}$$

for three types of differential operators H:

(I) Laplace–Beltrami operators $\Delta = \sum_{ij} \frac{1}{\sqrt{g}}\, \partial_i g^{ij} \sqrt{g}\, \partial_j$, on compact manifolds $\mathcal{M}$ with Riemannian metric $g_{ij}dx^i dx^j$ $[g = \det(g_{ij})]$;

(II) Laplacians Δ on bounded regions Ω in $\mathbb{R}^n$ (or more general manifolds) with suitable boundary conditions on $\Gamma = \partial\Omega$ (Dirichlet, Neumann, mixed $[(u - h\partial_n u)|_\Gamma = 0]$, periodic, etc.);

(III) Schrödinger operators with scalar and vector potentials: $H = -\Delta + V(x)$, or $(i\nabla + A)^2 + V$, on manifolds $\mathcal{M}$ or regions Ω.

In all three cases the operator H is well known to have a discrete real spectrum: $H[\psi_k] = \lambda_k\psi_k$, $\psi_k \in L^2$, $\mathrm{spec}(H) = \{\lambda_k\}_0^\infty \subset \mathbb{R}$, accumulating to $\{\infty\}$. This follows from the general elliptic theory, through compactness of the resolvent or semigroup, $R_\zeta = (\zeta - H)^{-1}$ or e^{-tH} (see [Ag], [Hö]).

Forty More Years of Ramifications: Spectral Asymptotics and Its Applications, ed. by S. A. Fulling and F. J. Narcowich, **Discourses in Mathematics and Its Applications**, No. 1, © Department of Mathematics, Texas A&M University, College Station, Texas, 1991.

The main objective in spectral theory of differential operators is relations between

Geometric data: ceofficients of H; shape of the boundary; metric, etc.

and

Spectral data: eigenvalues/eigenfunctions of H.

The Direct Problem asks one to find $\{\lambda_k\}$, or certain asymptotic/approximate characteristics of spec(H) (spectral invariants) in terms of the operator's geometric data, while the Inverse Problem is to recover the geometric data (or a part of it) from suitable spectral data.

Both problems are interesting from the mathematical standpoint and find numerous applications in physics. The physical models include vibration of elastic bodies, acoustics, electromagnetic waves and quantum mechanics.

For all such models (0.1) describes the *proper modes* of motion of the system (standing waves), $\{\psi_k\}$.

Indeed, the standard wave equation,

$$(i) \qquad u_{tt} = H[u],$$

describing vibration of elastic bodies (strings, membranes), as well as propagation of sound and light, and the Schrödinger equation of quantum mechanics,

$$(ii) \qquad i\phi_t = H[\phi],$$

can both be expanded in eigenfunctions of H,

$$(i) \qquad u = \sum a_k e^{\pm i\omega_k t}\psi_k(x) \text{with frequencies } \omega_k = \sqrt{\lambda_k},$$

$$(ii) \qquad \phi = \sum a_k e^{i\lambda_k t}\psi_k(x).$$

The eigenvalues $\{\lambda_k\}$ determine all the *proper frequencies* of vibration $\{\omega_k = \sqrt{\lambda_k}\}$ in the wave problem (i), while in quantum mechanics H represents the *Hamiltonian (energy)* of a system, $\{\psi_k\}$ its *bound states*, and $\{\lambda_k\}$ the corresponding (quantized) *bound state energy levels.*

Solution of the inverse problem yields certain geometric and physical parameters (such as shape of the region, coefficients of conductivity, etc.) from frequencies/eigenvalues.

M. Kac was using such physical lingo when he coined his famous phrase "*Shape of the drum*" for the inverse spectral theory. In his seminal paper [Ka] Kac was specifically interested in the inverse "membrane problem": *determination of the boundary of* Ω *from the spectrum of the Dirichlet* (*Neumann*) *Laplacian* Δ_Ω.

But after [Ka] the terminology stuck to other spectral problems, including the "*Shape of the* (*Riemannian*) *metric*", and the "*Shape of the potential*" for Schrödinger operators.

In our paper we shall briefly survey the history of the subject, and then outline some recent progress in all three "Shape Problems".

§1. Weyl Formula, "Phase Volume" Counting and Spectral Invariants

Weyl originated to a large extent the field of spectral theory of differential operators by his seminal work [We1] on asymptotic distribution of large eigenvalues for the Dirichlet Laplacians in regions $\Omega \subset \mathbb{R}^n$, $H = -\Delta_\Omega$.

Let $N(\lambda) = \#\{\lambda_k(H) \le \lambda\}$ denote the *counting function* of H, the number of eigenvalues below the level λ. In 1911 young Weyl, on the suggestion of Hilbert, set out to investigate the asymptotics of $N(\lambda)$ as $\lambda \to \infty$, and derived his celebrated formula,

$$N(\lambda) \sim \frac{B_n}{(2\pi)^n}|\Omega|\lambda^{n/2} + \dots, \tag{1.1}$$

where $|\Omega| = \mathrm{Vol}(\Omega), B_n =$ surface area of the unit ball in $\mathbb{R}^n$.

He also conjectured the next term the expansion to be proportional to the area of the boundary, $\partial\Omega$; *i.e.*,

$$N(\lambda) \sim C_o\lambda^{n/2} + C_1\lambda^{(n-1)/2} + o(\dots), \tag{1.2}$$

with coefficients $C_0 = \mathrm{Const} \times |\Omega|$ and $C_1 = \mathrm{Const} \times |\partial\Omega|$.

However, it took many years after Weyl and significant efforts of several mathematicians to establish the second term of (1.2) rigorously and to estimate the remainder [Kac, Hö, Se, Me, Iv].

In 1934 Carleman [Ca] extended (1.1) to Schrödinger operators, where

$$N(\lambda) \sim C_0 \int (\lambda - V)_+^{n/2} dx,$$

$(\lambda - V)_+$ meaning the positive part of the function $\lambda - V(x)$.

He also proposed to replace $N(\lambda)$ by certain of its averages, such as the Mellin, Laplace, and Cauchy–Stieltjes transforms. The first two are often called *zeta and theta functions* of H:

$$\begin{aligned} Z(s) &= \mathrm{tr} H^{-s} = \sum_{k=0}^{\infty} d_k \lambda_k^{-s} = \int \lambda^{-s} dN(\lambda); \\ \Theta(t) &= \mathrm{tr} e^{-tH} = \sum_0^{\infty} d_k e^{-\lambda_k t} = \int e^{-\lambda t} dN(\lambda); \\ F(\zeta) &= \mathrm{tr}(\zeta - H)^{-m} = \sum_0^{\infty} \frac{d_k}{(\zeta-\lambda_k)^m} = \int \frac{dN(\lambda)}{(\zeta-\lambda)^m} \text{ for some } m \geq 1. \end{aligned} \tag{1.3}$$

Here d_k is the multiplicity of λ_k.

The main advantage in passing to averages has to do with asymptotic and analytic properties of the functions $Z(s)$, $\Theta(t)$, $F(\zeta)$. Whereas $N(\lambda)$ is typically highly irregular (no lower order asymptotics could exist beyond the 2^{nd} Weyl term), the corresponding "averages" admit nice expansions to all orders.

The foremost is the "heat expansion" of Minakshisundaram and Pleijel [Mu, MP], elaborated and extended by many other authors [McS, Gi] (see references in [Cha]),

$$\Theta(t) = \mathrm{tr}\ e^{-tH} \sim t^{-n/2}\{b_0 + b_1 t^{1/2} + b_2 t + \cdots\}, \text{ as } t \to \infty. \tag{1.4}$$

The coefficients $\{b_j\}$ are obviously spectral invariants of H, called *heat invariants*. It turned out that $\{b_j\}$ carry some important geometric/topological information. The first two are obviously related to Weyl's coefficients[1] (1.2):

$$b_0 = \Gamma\left(\frac{n}{2}+1\right) C_0, \qquad b_1 = \Gamma\left(\frac{n}{2}\right) C_1,$$

---

[1] Since all three kernels $(\lambda - H)^{-m}$, e^{-tH}, and H^{-s} are related by the Mellin/Laplace transforms,

$$(\lambda - H)^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t(H-\lambda)} t^{s-1} dt,$$

asymptotic expansion of any one of them can be converted via Abelian/Tauberian Theorems (Karamata, Keldysh [Ti]) to the other. In particular, the Z-function has simple poles at $\{s_k = \frac{\dim - k}{2}\}$, whose residues are proportional to the heat invariants $\{b_k\}$.

and thus recover the volume $|\Omega|$ and the area $|\partial\Omega|$.

McKean and Singer [McS] found the higher invariants b_2 and b_3 for Laplacians on manifolds with boundary;

$$b_2 = \frac{1}{3}\left(\int_{\mathcal{M}} K - \int_{\partial\mathcal{M}} J\right).$$

Here K = trace(Ricci) means the scalar curvature of $\mathcal{M}$, while J = trace(2^{nd} fundamental form) denotes the mean curvature of the boundary $\partial\mathcal{M}$.

By Gauss's Theorem for 2-D surfaces, b_2 is the *Euler characteristic* of $\mathcal{M}$, $b_2 = 1-$ genus (#handles). Thus "the topology of $\mathcal{M}$ is audible".

In general, the heat invariant b_j is given by the integral of some universal polynomial expression in curvature and its covariant derivatives over Ω (or $\partial\Omega$) [Gi]. However, the computational complexity increases rapidly with j, so very few heat invariants are known explicitly [Gi, Sm].

Let us remark that the "heat method" (and "heat coefficients") provided a powerful tool in the Atiyah–Singer Index Theory for elliptic systems (complexes) on manifolds, including the de Rham and signature complexes, [ABP, Gi].

One still does not know whether the family $\{b_j\}$ determines the spectrum uniquely, *i.e.*, represents a complete set of spectral invariants.

In high dimensions ($D \geq 4$) there are known examples of continuous isospectral deformations of the metric [GW], of the boundary [BBe, Uru], and of the potential [Br], [DeG]. So the entire spectrum, let alone $\{b_j\}$, fails to uniquely determine the "geometry".

In low dimensions ($D = 2, 3$) the inverse problem is widely believed to be *spectrally rigid*, in the sense that isospectral classes of metrics, boundaries, or potentials are finite (modulo symmetries). Let us also remark that all "continuous" counterexamples in high dimensions seem to be quite exceptional, so the rigidity may still be the prevailing phenomenon generically. We shall discuss the metric problem in the next section.

To conclude let us mention some special families of "audible drums":

(i) Disks are distinguished among all other drums by the first two Weyl/heat invariants, via the standard isoperimetric inequality between the volume and the area,

$$|\partial\Omega| \geq \text{Const}|\Omega|^{\frac{n-1}{n}}.$$

This was observed by Kac [Ka].

(ii) For triangles (in fact, all polygons) Kac derived the third heat invariant, depending on angles (see [Ka, BS]). Thus, in the family of all triangles each individual is audible via $\{b_0; b_1; b_2\}$.

Other special regions include ellipses [GM], certain N-parameter families of ovals [MM], and more general convex symmetric (with respect to x- and y-axis) plane regions with real analytic boundary [Co2].

Remark: Weyl's formula (1.1) was extended to fairly general classes of elliptic (or subelliptic) differential and pseudodifferential operators $H = \mathcal{H}(x; i\nabla)$ on compact manifolds/domains and on $\mathbb{R}^n$ ([Ag, Hö2, TS, Sj, Gur1] to name a few authors). Here $\mathcal{H}(x;p)$ denotes the *principal symbol of* H, a classical observable on the phase space $\mathcal{P} = \mathbb{R}^{2n}$ (or the cotangent bundle $T^*\mathcal{M}$). Then the Weyl formula takes the form of the "*volume counting principle*",

$$N(H;\lambda) \sim (2\pi)^{-n}\mathrm{Vol}\{\mathcal{H}(x;p) \leq \lambda\}, \text{ as } \lambda \to \infty. \tag{1.5}$$

In the terminology of quantum mechanics the operator $H = \mathcal{H}(x; \frac{\hbar}{i}\nabla)$ represents the *quantized classical Hamiltonian* $\mathcal{H}(x;p)$. So (1.5) becomes a very general (albeit crude) form of the *Correspondence Principle* of quantum mechanics, which asserts close relations between the classical hamiltonian $\mathcal{H}$ on $\mathcal{P}$ and its quantum counterpart, the operator H on $L^2(\mathcal{M})$. Indeed, thinking of $\{\psi_k\}$ as bound states of the quantal system, with energy levels $\{\lambda_k\}$, formula (1.5) asserts that

$$\begin{aligned}&\#\{\textit{quantal states } \psi : \textit{energy}(\psi) \leq \lambda\}\\ \approx\ &\textit{Volume}\{\textit{classical states } (x,p) : \mathcal{H}(x,p) \leq \lambda\}.\end{aligned}$$

Though the volume counting principle (1.5) predicts correct asymptotics and has been rigorously proven in many cases, it falls short of being universal.

Recently a number of examples have been discovered [Si, Fe, Ro, So, Gur2] which exhibit nonclassical behavior of $\{\lambda_k\}$. The simplest prototype is the Schrödinger operator $H = -\Delta + |x|^\alpha |y|^\beta$ in $\mathbb{R}^2$. Here "volume counting" predicts infinite volume (hence continuous spectrum!) at all energies $\lambda \geq 0$, whereas spec(H) is in fact discrete and obeys a nonclassical asymptotics [Si].

Other interesting examples of nonclassical behavior arise in Schrödinger operators with pure magnetic (gauge) potentials, $H = (i\nabla + \mathbb{A})^2$, [Ta, Co3]. Once again the Weyl principle fails to predict discrete spectra for such operators with any $\mathbb{A}$. Yet for certain potentials $\mathbb{A}$, and the corresponding fields $\mathbb{B} = \nabla \times \mathbb{A}$ (magnetic bottles), spec(H) was shown to be discrete and to display nonclassical asymptotic behavior.

§2. The Correspondence Principle and Periodic Orbits

In the previous section we demonstrated a number of geometric/topological spectral invariants of differential operators H: volume, area of the boundary, Euler characteristic, etc., all of which provide particular manifestations of the correspondence principle. However, to gain a deeper insight into the structure of the spectrum and to approach the inverse problem, we need to look at other geometric/dynamical characteristics of the corresponding classical systems.

The relevant systems are hamiltonian (geodesic) flows on the phase space $\mathcal{P} \simeq T^*(\mathcal{M})$, defined by the classical Hamiltonian, the symbol $h = h(x,p)$ of operator H,

$$\phi_t\colon\ (x,p) \to \exp tX(x,p), \tag{2.1}$$

where $X = X_h = h_x \cdot \partial_p - h_p \cdot \partial_x$ denotes the Hamiltonian vector field of h.

Several dynamical characteristics of (2.1) have been found to play roles in spectral theory:

i) *Classical close trajectories and the related Poincaré numbers.*
ii) *Invariant Lagrangians* $\Lambda \subset \mathcal{P}$, *i.e.*, maximal isotropic (relative to the symplectic structure) submanifolds of $\mathcal{P}$, invariant under the flow (2.1). Natural examples include invariant tori of integrable Hamiltonians, $\Lambda = \bigcap_m \{I_m(x;p) = \ \text{Const}\}$, where $\{I_1; I_2 \dots I_n\}$ form a maximal Poisson-commuting family on $\mathcal{P}$, and $h = f(I_1; \dots I_n)$. But there are interesting examples of higher-genus invariant Lagrangians [RB].
iii) *"Quantized" closed caustics* (envelopes of rays) [Ke, KR].

All of these appear in the context of the "large eigenvalue problem" for operators $H = h(x; i\nabla)$, and also in the semiclassical setup, $H = h(x; i\hbar\nabla)$ with small Planck parameter. But they have somewhat different ranges of applicability.

The existence of invariant Lagrangians (or rather, nice "Lagrangian foliations" $\Lambda(\alpha_1; \dots \alpha_n)$) essentially amounts to integrability of h, hence is limited in scope. Such continuous families are then quantized by the EBK (Einstein–Brillouin–Keller) rules to produce a discrete set of approximate eigenvalues of H (*cf.* [MF], [BT]).

Closed orbits are more universal and appear in a greater variety of geometric settings. So we shall mostly discuss the role of closed orbits.

There are various physical (heuristic) arguments behind the correspondence principle for closed orbits [Gut, BB, Vo, AA], as well as mathematically rigorous results [Ch, DG, Ra, Do].

The basic principle asserts that "*the spectrum* $\{\lambda_k\}$ *of the operator* H *determines the length/period spectrum* $\{L_m\}$ (*the length of all closed orbits*) *of the classical system* (2.1)".

To state precise results we consider all closed orbits $\{\gamma\}$ (geodesics or bicharacteristics) along with their Poincaré maps[2] $\{P_\gamma\}$ and Morse indices $\sigma(\gamma)$, and denote by $L = \ell(\gamma)$ the length of γ. It turns out that for fairly general second order elliptic operators H (including Laplacians), $\operatorname{spec}(\sqrt{H})$ contains asymptotic "almost arithmetic" sequences of large eigenvalues, determined by L and P_γ. A typical result [Ch, DG, Ra] for *stable* γ (*i.e.*, P_γ equivalent to a direct sum of rotations by angles $\{\theta_1; \dots \theta_{n-1}\}$) states

Theorem. *For any tuple of integers* $\{m_1; \dots m_{n-1}\}$, $\operatorname{spec}(H)$ *contains a sequence of eigenvalues,*

$$\sqrt{\lambda_k} \sim \frac{2\pi}{L} k + \sum_1^{n-1} m_j \theta_j + \sigma(\gamma) \quad \text{as } k \to \infty. \tag{2.2}$$

Mathematical proofs of (2.2) are based either on the *wave equation method*, microlocal analysis of the wave group $U_t = \exp(it\sqrt{H})$ (the "forward" fundamental solution of the wave equation $u_{tt} - H[u] = 0$), or the *quasimodes* (approximate eigenfunctions, localized near γ) [Ra].

The key idea is to construct the wave kernel U_t and study its formal trace, as a distribution on $\mathbb{R}$,

$$\chi(t) = \operatorname{tr}\left(e^{it\sqrt{H}}\right) = \sum e^{it\sqrt{\lambda_k}}. \tag{2.3}$$

The distribution χ is made of terms that represent contributions of various closed orbits [Gut, Vo, DG],

$$\chi = \chi_0 + \sum \chi_\gamma.$$

The main singularity, the contribution of zero length (one-pointed) paths, has its Fourier transform expanded at $\{\infty\}$ as

$$\widehat{\chi}_0(\mu) \sim c_0 \mu^{n-1} + c_1 \mu^{n-2} + \cdots \quad \text{as } \mu \to \infty, \tag{2.4}$$

[2] The Poincaré map measures the "cross-sectional twist" of the flow (2.1) along the closed orbit.

the coefficients being related to the Weyl (heat) invariants[3] of H, or the residues of its zeta-function. All other singularities are located at precisely the period spectrum $\{T_m\}$ of σ. Furthermore,

$$(\chi - \chi_0)(t) \sim \frac{1}{2\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{\ell(\gamma) i^{\sigma(\gamma)}}{(t - \ell(\gamma^m)) \sqrt{\det(I - P_\gamma^m)}}. \tag{2.5}$$

The outer summation in (2.5) extends over primitive closed paths $\{\gamma\}$, while γ^m denotes the m^{th} iterate of γ.

Formula (2.5) describes the contribution of isolated non-degenerate periodic orbits. In degenerate cases, when orbits of a given period form a cell (submanifold) of dimension d, more singular distributions of order $\frac{d-1}{2}$ appear in place of $\frac{1}{t-\ell(\gamma)}$ [DG].

We shall illustrate (2.5) by two well known examples:

1° The Laplacian on the torus $\mathbf{T}^n = [0; T_1] \times \cdots \times [0; T_n]$, with basic periods $\{T_1; \dots T_n\}$. Here the eigenvalues are known exactly: $\lambda_k = 4\pi^2 \sum (k_j/T_j)^2$, k running over all n-tuples $(k_1; \dots k_n) \in \mathbb{Z}^n$.

We apply the *Poisson summation* formula to the RHS of (2.3), considered as a function on the lattice $\Gamma = \operatorname{span}\{\frac{2\pi}{T_j} e_j\}$ in $\mathbb{R}^n$:

$$\sum_{k \in \Gamma} f(k) = \text{ Vol } \sum_{m \in \Gamma'} \widehat{f}(m) \tag{2.6}$$

---

[3] Formula (2.4) can be formally derived from Weyl's asymptotics, $\lambda_k \sim 4\pi^2 (\frac{k}{\text{Vol}})^{n/2}$. Indeed, pairing the distribution χ to a test function f, and applying (1.1), we get

$$\begin{aligned} \langle \chi; f \rangle &= \sum \widehat{f}\left(\sqrt{\lambda_k}\right) \simeq \sum \widehat{f}\left(2\pi \left(\frac{k}{V}\right)^{\frac{1}{n}}\right) + \cdots \simeq \int^{\infty} \widehat{f}(\dots)\, dk \\ &= \frac{n\text{Vol}}{(2\pi)^n} \int^{\infty} \widehat{f}(r) r^{n-1} dr + \cdots . \end{aligned}$$

By the Plancherel identity $\langle \chi; f \rangle = \langle \widehat{\chi}; \widehat{f} \rangle$, so we get (2.4). Of course, in practice (2.4) is derived via "microlocal analysis" of the wave kernel $U_t(x; y)$, and then applied to prove (1.1) [Hö1].

for a nice function/distribution f on $\mathbb{R}^n$ and its Fourier transform $\widehat{f}$. Here Γ' denotes the dual lattice, and Vol refers to the volume of the fundamental region $\mathbb{R}^n/\Gamma'$.

Specializing (2.6) to a function $f(x) = e^{it|x|}$ on Γ, with Fourier transform

$$\widehat{f}(\xi) = C_n t^{-\frac{n-1}{2}} \delta^{\left(\frac{n-1}{2}\right)}(t - |\xi|)$$

on the dual lattice $\Gamma' = \operatorname{span}\{T_j e_j\}$, we get

$$\chi(t) = C_n \mathrm{Vol}\ t^{-n/2} \sum_m \delta^{(\dots)}(t - L_m), \tag{2.7}$$

with $L_m = \sqrt{\sum (m_j T_j)^2}$ = the length of the m^{th} orbit in $\mathbf{T}^n$. Formula (2.7) clearly demonstrates the "length spectrum" singularities of the wave-kernel.

2° Another well known example is the celebrated *Selberg trace formula* for Laplacians on compact hyperbolic surfaces $\mathcal{M} = \mathbb{H}/\Gamma$, the Poincare half-plane $\mathbb{H} = \{z : \Re z > 0\}$ modulo a discrete subgroup Γ of $SL(2;\mathbb{R})$, acting by fractional linear transformations on $\mathbb{H}$.

This time we shall rephrase the result in terms of the heat (rather than wave) expansion. Singularities of the wave trace χ would then correspond to various exponential terms of $\Theta(t)$ (see [DG]).

The closed paths of $\mathcal{M}$ are labeled by conjugacy classes $\{k^{-1}gk\colon\ k \in \Gamma\}$ of elements g in Γ, each class contains a power p^n of a hyperbolic primitive element $p \sim \begin{pmatrix} m & \\ & m^{-1} \end{pmatrix}, m > 1$. We denote by $\ell(g) = \cosh^{-1}\left(\frac{1}{2}\ \mathrm{tr}|g|\right)$ the length of the closed path $\gamma = \gamma(g)$ (see [Mc]), where $|g| = (g^* g)^{1/2}$. Then one has the following expansion of the theta-function of the hyperbolic Laplacian on $\mathcal{M}$,

$$\Theta(t) = \frac{e^{-t/4}}{(4\pi t)^{3/2}} \left\{ C(t)|\mathcal{M}| + \pi t \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjug} \\ \text{primit} p}} \frac{\ell(p)}{\sinh \frac{1}{2}\ell(p^n)} e^{-\ell(p^n)^2/4t} \right\}. \tag{2.8}$$

The summation (2.8) extends over all powers p^n of primitive elements (repeated geodesics), which label the conjugacy classes in Γ, $|\mathcal{M}|$ denotes the volume (area) of manifold $\mathcal{M}$, in accord with (1.1)—(1.2), and $C(t)$ is a universal coefficient

$$C(t) = \int_0^\infty \frac{x e^{-x^2/4t}}{\sqrt{\cosh x - 1}} dx.$$

Once again the length spectrum $\{L_n = \ell(p^n)\}$ appears explicitly in the expansion (2.8), so it can be thought of as a noncommutative version of the Poisson summation (2.6).

Let us remark that $C(t) \sim \sqrt{4\pi t}$ for small t, which makes (2.8) consistent with the well known local behavior of the heat kernel $K_t(x;y)$ on Riemannian manifolds [Cha]:

$$K_t(x,y) \sim (4\pi t)^{-n/2} e^{-d^2(x;y)/4t} + \cdots, \quad \text{for small } t \text{ and } d(x;y),$$

where $n = \dim \mathcal{M}$ and $d(x;y)$ is the distance between x, y in $\mathcal{M}$. So the leading asymptotic of $\Theta(t)$ in (2.8) becomes $\frac{1}{4\pi t}e^{-d^2/4t}$, as befits a 2-D surface!

Formulae (2.6) and (2.8) closely resemble the asymptotic expansion (2.5), but the remarkable feature of the torus and of the hyperbolic Laplacian on H/Γ is that the quasiclassical relations between "spectra" and "geometry/dynamics" become exact!

The standard physical arguments for (2.5) and (2.8) are based on the Feynman–Kac path-integral representation of the Green's function of H (the resolvent or semigroup kernel) as the "sum over all paths in $\mathcal{P}$" of the classical action $S = \oint \mathcal{L}(x;\dot{x})dt$, where $\mathcal{L} = p \cdot \dot{x} - h$ is the Lagrangian of h. By the stationary-phase principle the most important contribution to the Feynman integral comes from the "critical points", *i.e.*, the *closed classical paths*. Applying such principles to the heat (and related) kernels one can get formal *asymptotic expansions* like (2.5), but it is not clear how the path integral method could justify *exact expansions* like (2.6) and (2.8).

For more details and motivation we refer to [BB, Gut, Vo, AA].

Returning to the inverse problem, the length spectrum of closed paths (geodesics) reduces it to a problem in geometry/dynamics on manifolds: *Given two metrics on $\mathcal{M}$ whose geodesic flows have identical period/length spectra, can one conclude that those metrics are isometrically equivalent?*

In the classical case of hyperbolic surfaces $\mathcal{M} = \mathsf{H}/\Gamma$ with constant negative curvature, McKean [Mc] showed that there are at most finitely many isospectral surfaces $\{\mathcal{M}\}$. He utilized Selberg's trace formula (2.8) and the Klein–Fricke double and triple traces $\{\mathrm{tr}(\gamma_k\gamma_m),\ \mathrm{tr}(\gamma_i\gamma_k\gamma_m) : \gamma_k \in \Gamma\}$. The latter served to identify elements $\{\gamma_k\}$ up to conjugacy in Γ, hence the lengths of closed geodesics $\{\ell(\gamma)\}$ and their orientations.

McKean's result [Mc] left a possibility of *uniqueness* in the inverse problem on hyperbolic surfaces H/Γ, until Vigneras [Vi] found in 1980 finite (but arbitrarily large!) families of distinct isospectral surfaces in any genus. These examples, however, turned out to be rather exceptional. S. Wolpert [Wo] showed that *a generic hyperbolic surface is uniquely determined by* spec(Δ)!

For more general hyperbolic manifolds of non-constant curvature Guillemin and Kazhdan [GK] established "*infinitesimal rigidity*": Any continuous 1-parameter family of isospectral deformations must be trivial. Their method also used a reduction to a dynamical problem.

In the study of dynamical problems it was found that *uniqueness* usually requires an additional "discrete set of data", for instance, association of homotopy classes to the periods $\{L_m\}$ (see [KB]). The latter had previously been used by McKean [Mc] in the constant-curvature case. Under such additional hypotheses, the geometric dynamical problem was shown to have a unique solution [KB].

So one comes tantalizingly close to settling the inverse spectral question (at least on hyperbolic surfaces), but there remains the frustrating "labeling problem", whose spectral content is unclear. The best results to date [OPS] establish compactness of isospectral metrics in the $\mathcal{C}^\infty$ topology, as well as compactness of shapes (boundaries) of 2-D drums. It exploits a novel approach, based on determinants of Laplacians. This approach was further extended in [CY] to some 3-D metric problems.

§3. The Shape of the Potential

In this last section we turn to Schrödinger operators $H_V = -\Delta + V$ on compact manifolds/domains, and study the relation between spec(H_V) and V.

Let us emphasize that we are interested in Schrödinger operators with large Planck constant, $\hbar = 1$, as opposed to the more familiar semiclassical setting with small $\hbar$, $H_\hbar = -\hbar^2\Delta + V$. The semiclassical problem can be reduced to a "large potential" problem via rescaling, $H_\hbar = \hbar^2(-\Delta + \hbar^{-2}V)$, hence the kth "semiclassical" eigenvalue is essentially the kth "large potential" eigenvalue:

$$\lambda_k(V, \hbar) = \hbar^2 \lambda_k(\hbar^{-2}V).$$

So the difference between the two problems lies in the different asymptotic ranges of the eigenvalue/energy parameter $\lambda(V, \hbar)$. Namely, for moderate energies, $\lambda(V, \hbar) = \mathcal{O}(1)$, *i.e.*, $\lambda_k(\hbar^{-2}V) = \mathcal{O}(\hbar^{-2})$, the underlying classical dynamics is dominated by the complete Hamiltonian $h_V = p^2 + V$, as both terms, kinetic energy $P^2 = (i\hbar\nabla)^2$ and potential energy V, are of equal strength. Hence the relevant classical data entering the expansions (2.5) are closed orbits (bicharacteristics, rays) of h_V.

On the other hand, for large energies, $\lambda(\hbar) = \mathcal{O}(\hbar^{-2})$, or, equivalently, large-eigenvalue asymptotics of H_V, the kinetic term $H_0 = P^2 = -\Delta$ dominates, so H_V can be considered as a regular perturbation of H_0. The underlying classical dynamics is governed by the free Hamiltonian $h_0 = p^2$, so it coincides with the geodesic (billiard ball) flow on $\mathcal{M}$, and the relevant dynamic data consists of closed geodesics.

In the latter case V enters various eigenvalue asymptotic formulae in the form of integrals $\oint_\gamma V$ along closed paths, rather than "classical orbit perturbations".

The role of the length spectrum will be played now by the *Radon transform* of V,

$$V(x) \to \widetilde{V}(\gamma) = \oint_\gamma V\, ds, \quad \text{integral over a closed geodesic } \gamma. \tag{3.1}$$

A heuristic explanation for (3.1) comes once again from the Kac–Feynman path-integral representation of the heat kernel of H_V,

$$e^{t(\Delta - V)}(x;x) = \int d\mathcal{B}_t[\gamma] e^{-\oint_\gamma V ds}.$$

Here $d\mathcal{B}_t[\gamma]$ denotes the Wiener integral over the so-called *Brownian bridge*: the set of all closed paths, $\gamma = \gamma_t(x)$, connecting $\{x\}$ to itself in time t [Mo, AA]. Formal application of the "stationary phase/steepest descent" method keeps the contribution of the *stationary* (*classical*) *path* only, whence comes (3.1).

Therefore, the Schrödinger Inverse Problem could be approached in two basic steps:

(i) *derivation of the Radon transform* $\widetilde{V}$, *or some "functionals" of* $\widetilde{V}$, *from spectral asymptotics of* H;
(ii) *the proper Radon transform inversion procedure.*

The classical and best studied example of inverse problems are 1-D regular Sturm–Liouville (S-L) operators: $H = -\partial^2 + V(x)$ on $[0,1]$, with various types of boundary conditions: 2-point, periodic, Floquet, etc.

Such an operator is known to have a multiplicity-free spectrum $\{\lambda_k\}$, which admits the following asymptotic expansion [Bo, Le],

$$\lambda_k = (\pi k)^2 + b_0 + k^{-1} b_1 + \cdots \quad \text{as } k \to \infty. \tag{3.2}$$

Here $b_0 = \int V dx$, and the higher b_j are expressed in terms of Fourier coefficients of $\tilde{V}$.

The correspondence $V \to \mathrm{spec}(H_V)$ for S-L operators proves to be highly non-unique: there are typically large (infinite-dimensional) isospectral classes both in the periodic/Floquet [La, No, MM] and the "2-point boundary" case [IMT, PT]. So a unique determination of V requires an additional infinite set of data, such as the "KdV-flow parameters", "norming constants", etc. These results extend to some singular S-L problems, like the harmonic oscillator [MT], and operator $H = -\partial^2 + (1/x^2)$ on $[0,1]$, [GR].

Compared to the 1-D case, multivariable Schrödinger theory is far less developed and understood. Moreover, multi-dimensional Schrödinger operators often exhibit different qualitative features, like *spectral rigidity*, proposed by V. Guillemin. Namely, the isospectral classes, $\mathrm{Iso}(V)$, consist of finite-dimensional families obtained by natural (geometric) symmetries (isometries of $\mathcal{M}$), rather than the hidden "KdV-type symmetries".

The *Rigidity Hypothesis* has been confirmed in a number of cases. The foremost is the case of negatively curved manifolds [GK], where, under some additional assumptions (simple "length spectrum" of the geodesic flow), it was shown that V *is uniquely determined by* $\mathrm{spec}(H_V)$, so $\mathrm{Iso}(V) = \{V\}$. Of course, negatively curved manifolds typically have no continuous internal (geometric) symmetries. In this respect the potential problem on hyperbolic spaces proved to be more manageable than the metric problem (§2).

The main tool of [GK] was to show that the Radon transform of V is determined by spectral asymptotics of H_V. This data combined with some specific features of hyperbolic geometry (discrete, but rapidly proliferating with length, families of closed geodesics) proved to be sufficient to recover V.

Another well studied case is the flat torus $\mathbb{T}^n$. Its symmetries consist of all translations and reflections. It was shown that *generic analytic potentials*[4] V *on* $\mathbb{T}^n$ *are spectrally rigid* [ERT, MN], and there is strong evidence that this holds for nonanalytic potentials as well. The argument was based on certain reduced S-L operators and involved once again the Radon transform of V.

The case of the positively curved (and highly symmetric) n-sphere turned out to be the most difficult. So far only partial results have been obtained [Wei, Gui, Co, Ur, Gur4].

In the rest of the paper we shall outline the state of the n-sphere theory.

---

[4] Obviously, special potentials, such as $V = V_1(x_1) + \cdots + V_n(x_n)$, the sum of 1-variable functions, may still possess "large" (infinite) iso-classes, obtained by the hierarchy of KdV-flows applied to each 1-D constituent $\{V_j(x_j)\}$.

The n-sphere Laplacian has regularly distributed and highly degenerate spectrum: $\mathrm{spec}(-\Delta) = \{\lambda_k = k(k+n-1)\}$, the multiplicity of λ_k increasing rapidly with k, $d_k = O(k^{n-1})$. The degeneracy evidently results from the underlying rotational symmetry $SO(n+1)$.

Adding the potential V breaks the rotational symmetry, so $\mathrm{spec}(H_V)$ splits into clusters of simple (less degenerate) eigenvalues $\Lambda_k = \{\lambda_{km} = \lambda_k + \mu_{km}: m = 1, \dots d_k\}$. The clusters are asymptotically well localized: the cluster size being

$$|\Lambda_k| = \begin{cases} \mathcal{O}(1), & \text{for even/generic } V, \\ \mathcal{O}(k^{-2}), & \text{for odd } V, \end{cases}$$

while the distance between neighboring Λ_k increases in proportion to k [Gui2].

So it is natural to look for spectral invariants associated with the cluster structure of $\mathrm{spec}(H)$.

To find such invariants A. Weinstein [We] introduced a sequence of *cluster-distribution measures*,

$$d\nu_k = \frac{1}{d_k} \sum_m \delta(x - \mu_{km}).$$

He proved that the sequence $\{d\nu_k\}$ converges to a continuous measure $\beta_0(\lambda)d\lambda$ on $\mathbb{R}$, whose density is equal to the distribution function of (not surprisingly!) the *Radon transform* $\widetilde{V}$, $d\nu_k \to \beta_0 d\lambda$, where

$$\langle f; \beta_0 \rangle = \int_{\mathcal{O}} f \circ \widetilde{V}\, dS, \quad \text{for any test function } f \text{ on } \mathbb{R}, \tag{3.3}$$

integration over the space $\mathcal{O}$ of all closed geodesics (great circles)[5] on S_n.

Moreover, by analogy with (3.2), $\{d\nu_k\}$ can be expanded in powers of k^{-1},

$$d\nu_k \sim \beta_0 + k^{-1}\beta_1 + \cdots, \tag{3.4}$$

where the coefficients $\{\beta_0; \beta_1 \dots\}$ represent certain distributions on $\mathbb{R}$ depending on V. Weinstein called the $\{\beta_k\}$ *band-invariants*, and calculated β_0 (3.3).

---

[5] Let us remark that the space of closed geodesics $\mathcal{O}$ on S_n is very different from that in the hyperbolic case—a discrete sequence of isolated paths $\{\gamma_m\}$—or the flat (torus) case—a discrete union of cells $\mathcal{O}_L = \{\gamma : |\gamma| = L\} \simeq \mathbb{T}^{n-1}, L = (\sum m_j^2)^{1/2}$. The spherical-space $\mathcal{O}$ consists of a single "fat" cell $\mathcal{O} \simeq S^*(S_n)/\mathbb{T}$, of dimension $2(n-1)$, which possesses many other structures (homogeneous space of $SO(n+1)$, complex projective variety) [Gui, Ur]. The entire geodesic flow is periodic of period 2π.

The higher band-invariants turned out much harder to calculate explicitly; only two of them $\{\beta_1;\beta_2\}$ have been found so far [Ur1—2, Gur].

At first glance Weinstein's result seemed to provide a crucial link to the inverse problem: following the general strategy, one only needs to "invert" the transform, $\beta_0 \to V$, to recover V (modulo rigid motions).

The main difficulty, however, lies in the fact that the distribution function of $\widetilde{V}$ does not determine $\widetilde{V}$ itself, even in the 1-variable case, whereas the S_n-Radon transform depends generically on $2(n-1)$.

Of course, the inverse problem would be immediately solved if we knew $\widetilde{V}$, rather than its distribution function.

Nevertheless, V. Guillemin [Gui1] successfully applied β_0 along with the classical heat invariants to prove spectral rigidity for special classes of potentials: low-degree spherical harmonics on S_2.

In our recent work [Gur4] we have studied another class, the so-called *zonal potentials* V; *i.e.*, functions $\{V\}$ which depend on a single azimuthal angle ϕ, or $x = \cos\phi$.

The corresponding Schrödinger operators possess an additional symmetry: on S_2 they commute with the z-component of the angular momentum operator $M = i\partial_\theta$, while on S_n the role of M is played by the entire Lie algebra $so(n)$ of axisymmetric rotations. Hence one can study the joint spectrum of H and M. In other words, the spectral shifts $\{\mu_{km}\}$ of the kth cluster acquire an additional (bigraded) structure, with an index m labeling the angular momentum of the corresponding eigenfunction ψ_{km}:

$$H[\psi_{km}] = (\lambda_k + \mu_{km})\psi_{km}, \quad M[\psi_{km}] = m\psi_{km}. \tag{3.5}$$

In this context we made an important improvement over the Weinstein result (3.5), by replacing asymptotics of cluster-distribution measures $\{d\nu_k\}$ by asymptotics of individual spectral shifts $\{\mu_{km}\}$.

To state the result we shall introduce three linear operators on $[0,1]$-functions, identified with zonal potentials $\{V(x)\}$:

(i) the reduced (zonal) Radon transform[6] $\mathfrak{R}$ (see [Gur4])

$$\mathfrak{R} : f(x) \to \tilde{f}(r) = \frac{2}{\pi}\int_0^{\sqrt{1-r^2}} f(x)\frac{dx}{\sqrt{1-r^2-x^2}};$$

---

[6] Let us remark that the Radon transform of a zonal function is also zonal, *i.e.*, depends on a single variable $x \in [0,1]$.

(ii) the Gegenbauer–Legendre operator (reduced Laplacian)

$$\mathcal{L} = \frac{d}{dx}(1-x^2)\frac{d}{dx} + \left(\frac{\alpha x}{1-x^2}\right)^2 - \alpha, \text{ where } \alpha = \frac{n-2}{2};$$

(iii) the Euler operator $\mathfrak{E} = x\frac{d}{dx}$.

Theorem 3 ([Gur4]). *The spectral shifts $\{\mu_{km}\}$ of the joint H, M-eigenvalue problem (3.5) admit an asymptotic expansion, similar to (3.2),*

$$\mu_{km} \sim a\left(\frac{m}{k}\right) + k^{-1}b\left(\frac{m}{k}\right) + k^{-2}c\left(\frac{m}{k}\right) + \cdots,$$

with certain function-coefficients $a(x), b(x), c(x)$, depending on $V(x)$. The coefficients $\{a,b,c\}$ are computed explicitly in terms of operators $\mathfrak{R}, \mathfrak{L}$ and $\mathfrak{E}$, applied to the even and odd parts of the potential $V = V_{\text{ev}} + V_{\text{od}}$:

$$\begin{aligned} a(r) &= \mathfrak{R}(V_{\text{ev}}) = \widetilde{V}_{\text{ev}}(r), \\ b(r) &= -\frac{1}{4}\mathfrak{R}\mathfrak{L}(V_{\text{ev}}), \\ c(r) &= \frac{1}{32}\mathfrak{R}\{\mathfrak{L}^2 + 2(n+1)\mathfrak{L}\}(V_{\text{ev}}) + \\ &+ \frac{1}{2}\{\mathfrak{R}(V_{\text{ev}}^2 + V_{\text{od}}^2) - (\mathfrak{R}V_{\text{ev}})^2\} - \frac{r^2}{2(1-r^2)}\left\{\mathfrak{R}\mathfrak{E}(V_{\text{ev}}^2 + V_{\text{od}}^2)\right\}. \end{aligned}$$

In the special cases of even and odd zonal potential V on S_2, we get

$$\mu_{km} = \begin{cases} \widetilde{V}\left(\frac{m}{k}\right) + O(k^{-1}) & \text{for even } V, \\ k^{-2}U\left(\frac{m}{k}\right) + O(k^{-3}) & \text{for odd } V, \end{cases} \tag{3.6}$$

where the function $U(x)$ is obtained from V_{od}^2 by the $\frac{1}{4}(\mathfrak{R} - \frac{r^2}{1-4^2}\mathfrak{R}\mathfrak{E})$-transform.

Theorem 3 exhibits a multiscale structure of spec(H), with the largest scale, $\mathcal{O}(k^2)$, locating clusters on the real line, a lower scale, $\mathcal{O}(1)$, indicating the cluster size, and yet smaller scales, $\mathcal{O}(k^{-1})$, etc., determining the distribution of spectral shifts within clusters.

As a corollary of Theorem 3 we got a unique and explicit solution of the inverse problem for the joint (H,M)-spectrum; established *local spectral rigidity* for generic zonal potentials (thus providing a partial answer to Guillemin's rigidity hypothesis); and applied (3.6) to calculate the first 3 Weinstein band-invariants for zonal operators.

The techniques of the n-sphere Schrödinger theory rely heavily on spherical symmetries and the ensuing 2π-periodicity both on the classical level (geodesic flow) and on the quantum (periodic spectrum of the Laplacian).

The basic steps involve a suitable *averaging procedure* [We, Gui, Ur], *symbolic calculi* on S_n [Ur], and in the zonal case [Gur4] a so-called *zonal reduction.* The underlying mechanism for rigidity, discovered in [Gur4], was due to "rigidity of almost arithmetic sequences with incommensurate differences".

Let us remark that some results of the S_n theory extend to rank-1 symmetric spaces.

But there remain many more open problems on both S_n and more general manifolds and operators. We shall list a few of them.

(i) Higher rank spaces and manifolds would require a new form of "multiparameter averaging" and a better insight into the role of closed orbits in the correspondence principle. It has been suggested that "geodesically flat tori" could play the role of closed geodesics here.

(ii) Generic (nonzonal) potentials and manifolds: their analysis could also be facilitated by the new "averaging methods."

(iii) Interesting examples of Hamiltonians are given by completely integrable systems (Calogero–Moser, Toda, etc.); these are known to be integrable on both classical and quantum levels [OP]. But very little is known about perturbations of integrable Hamiltonians, except some 1-D examples, such as perturbations of the harmonic oscillator [MT, Gur3].

(iv) Operators on vector bundles, as well as Hamiltonians with vector (magnetic) potentials, and elliptic systems. There is a number of direct spectral results, which show the effects of holonomy, Chern classes, and other geometric/topological data on spec(H) (cf. [Ku, GU, Co3]). But the inverse problem is not attempted yet.

In our brief survey we have been able to outline only a few of the exciting developments that the spectral theory of differential operators has undergone in the forty years since Weyl's "Ramifications". This progress will undoubtedly continue in the future, and bring a steady tide of fresh ideas, methods, problems and fruitful connections between physics and mathematics.

References

[AA] S. Albeverio and T. Arede, *The relation between quantum mechanics and classical mechanics: a survey of some mathematical aspects*, in Chaotic Behavior in Quantum Systems, G. Casati, ed., Plenum Press 1985, 37–76.

[Ag] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, van Nostrand, 1964.

[ABP] M. Atiyah, R. Bott, and V. Patodi, *On the heat equation and the index theorem*, Inv. Math. 19 (1973), 279–330.

[BB] R. Balian and C. Bloch, Ann. Phys. 64 (1971), 271–307; Ann. Phys. 69 (1972), 76–160; Ann. Phys. 85 (1974), 514–545.

[BT] M. Berry and M. Tabor, *Closed orbits and the regular bound spectrum*, Proc. R. Soc. Lond. A. 349 (1976), 101–123.

[BBe] Berard and G. Besson, *Spectres et groupes cristallographiques, I: Domaines euclidiens*, Inven. Math., 58 (1980), 179–199; *II: Domaines spheriques*, Ann. Inst. Fourier, Grenoble, 30 (1980), 237–248.

[BS] M. van der Berg and S. Srisatkunarajan, *Heat equation in a region in* $\mathbb{R}^2$ *with a polygonal boundary*, J. London Math. Soc. 2, 37 (1988), 119–127.

[Bo] G. Borg, *Umkerhrung der Sturm–Liouvillischen Eigenwertaufgabe Bestimmung der differentialgleichung die Eigenwerte*, Acta Math. 78 (1946), 1–96.

[Br] R. Brooks, *On manifolds of negative curvature with isospectral potentials*, Topology 26, 1 (1987), 63–66.

[Ca] T. Carleman, *Sur la theorie mathematique de l'equation de Schrödinger*, Ark. Mat. Astr. Fys. 24, 11 (1934), 34–44.

[Ca] *Chaotic Behavior in Quantum Systems*, G. Casati, ed., NATO ASI Series, v.120, Plenum Press 1985.

[Cha] I. Chavel, *Eigenvalues in Riemannian Geometry*, Pure and Appl. Math., vol. 115, Acad. Press, 1985.

[Ch] J. Chazarain, *Formule de Poisson pour les varietes Riemanniens*, Inv. Math. 24 (1974), 65–82.

[CY] A. Chang and P. Yang, *Isospectral conformal metrics on 3-manifolds*, preprint (1989).

[Co] Y. Colin de Verdiere, 1) *Spectre conjoint d'operateurs pseudodifferentials qui commutent*, Math. Z, 171, 51–73 (1980).

2) *Sue les longuers des trajectoires periodic d'un billiard*, Seminaire Sud–Rhodanien de Geometrie 111 (1983), 122–140.

3) *L'asymptotique de Weyl pour les bouteilles magnetiques*, Com. Math. Phys. 105 (1986), 327–335.

4) *Inverse spectral problem for the 2-sphere Schrödinger operators with zonal potentials*, Letters Math. Phys. 16 (1988), 313–323; *2-sphere Schrödinger operators with odd potentials*, Inverse Problems, (1990), 371–378; *Zonal Schrödinger operators on the n-sphere: Inverse spectral problem and rigidity*, Com. Math. Phys., (1990).

[Hö] L. Hörmander, 1) *The spectral function of an elliptic operator*, Acta Math. 121 (1968), 193–218.

2) *On the asymptotic distribution of eigenvalues of pseudodifferential operators*, Ark. Math. 17 (1979), 297–313.

[IMT] E. Isaacson, H. P. McKean, and E. Trubowitz, *The inverse Sturm–Liouville problem, I, II*, Comm. Pure Appl. Math. 36 (1983), 767–783; 37 (1984), 1–11.

[Iv] V. Ivrii, *Second term of the spectral asymptotic expansion of the Laplace–Beltrami operator on manifolds with boundary*, Funct. Anal., Appl. 14 (1980), 98–106.

[Ka] M. Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly 73, (1966), 1–23.

[KB] A. Katok and K. Burns, *Manifolds with nonpositive curvature*, Ergod. Th. of Dyn. Sys. 5 (1985), 307–317.

[Ke] J. B. Keller, *Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems*, Ann. Phys. 4 (1958), 180–188.

[KR] J. B. Keller and S. I. Rubinow, *Asymptotic solution of eigenvalue problems*, Ann. Phys. 9, 1 (1960), 24–75.

[KF] F. Klein and R. Fricke, *Vorlesungen über die Theorie der Elliptischer Modularfunktionen/Automorphenfunctionen*, G. Teubner, Leipzig, 1896/1912.

[Ku] R. Kubovara, 1) *Spectrum and holonomy of the line bundle over the sphere*, Math. Zeitschrift 187 (1984), 481–490.

2) *Spectrum of the Laplacian on vector bundles over $C_{2\pi}$-manifolds*, Diff. Geom. (1988), 241–258.

[La] P. Lax, *Periodic solutions of the KdV problem*, Com. Pure Appl. Math. 28 (1975), 141–188.

[Le] B. M. Levitan, *The Inverse Sturm–Liouville Problem*, Moscow, Nauka, 1984.

[MF] V. Maslov and M. Fedoryuk, *Quasiclassical Approximations for the Equations of Quantum Mechanics*, Moscow, Nauka, 1976.

[MM] S. Marvizi and R. Melrose, *Spectral invariants of convex planar regions*, J. Diff. Geom. 17 (1982), 475–502.

[McS] M.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. 1 (1967), 43–69.

[So] M. Solomyak, *Spectral asymptotics of Schrödinger operators with nonregular homogeneous potentials*, Mat. Sbornik, 127 (1985), 21–39.

[Ta] H. Tamura, *Asymptotic distribution of eigenvalues for Schrödinger operators with magnetic fields*, Nagoya Math. J. 105 (1987), 49–69.

[Ti] E. Titchmarch, *Eigenfunction Expansions Associted with Second-order Differential Equations*, II, Oxford, 1958.

[TS] V. Tulovskii and M. Shubin, *On asymptotic distribution of eigenvalues of pseudodifferential operators in* $\mathbb{R}^n$, Math. USSR (Sbornik) 21 (1973), 17–39.

[Ur] H. Urukawa, *Bounded domains which are isospectral but not congruent*, Ann. Scien. Ecole Norm. Sup. 15 (1982), 441–456.

[Vo] A. Voros, 1) *The WKB–Maslov method for nonseparable systems*, CNRS Symposium on Symplectic Geometry and Math. Physics, 1974.

2) *Unstable periodic orbits and semiclassical quantization*, J. Phys. A (1989).

[Ur] A. Uribe, 1) *A symbol calculus for a class of pseudodifferential operators on* S_n, J. Funct. Anal. 59 (1984), 535–556.

2) *Band invariants and clased trajectories on* S_n, Adv. Math. 58 (1985), 3.

[Vi] M. Vigneras, *Varietés riemanniennes isospectrales et non isometriques*, Ann. Math. 112 (1980), 21–32.

[Wei] A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J. 44 (1977), 883–892.

[Wey] H. Weyl, 1) *Über die asymptotische verteilung der Eigenwerte*, Gott. Nach. (1911), 110–117.

2) *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller differentialgleichungen*, Math. Ann. 71 (1911), 441–479.

3) *Ramifications, old and new, of the eigenvalue problem*, Bull. AMS 56 (1950), 115–139.

[Wo] S. Wolpert, 1) *The eigenvalue spectrum as moduli for compact Riemann surfaces*, Bull. AMS 83 (1977), 1306–1308.

2) *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces*, preprint (1986).