Reaction - diffusion in 1D

D. GURARIE

$$\begin{cases} u_t = \underbrace{D u_{xx}}_{\text{diff}} + \underbrace{f(u)}_{\text{react}} ; \quad BC \ldots \end{cases}$$

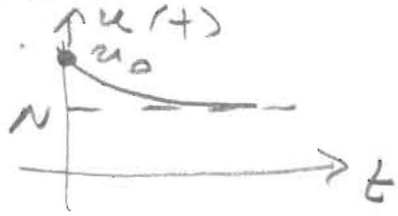
Examples:

1) $\underbrace{A}_{u} + \underbrace{B}_{v} \xrightarrow{k} \text{product} \Rightarrow \begin{cases} \dot u = -kuv \\ \dot v = -kuv \end{cases} \quad \begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

$u ; v$ - concentr.

Mass conserv. $\Rightarrow u - v = u_0 - v_0 = N$ - const

$\Rightarrow$ Logistic DE: $\boxed{\dot u = kN \underbrace{\left(1 - \frac{u}{N}\right) u}_{f}}$



2) FK model (slow evolution of allele freq.)

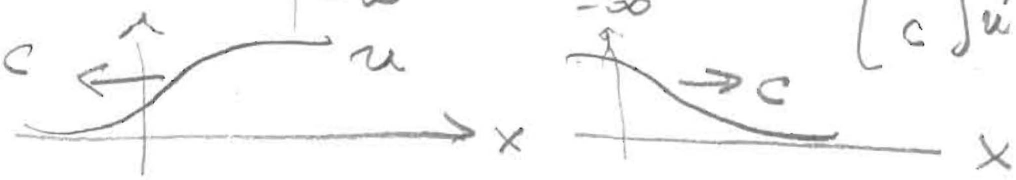
$$\boxed{p_t = D p_{xx} + a p (1-p)}$$

I. Equilibria: $\underset{u(x)}{}$ $\begin{cases} u'' + f(u) = 0 \\ BC \ldots \end{cases}$

II. Traveling waves: $u(x - ct)$ $\begin{cases} u'' + cu' + f(u) = 0 \\ u|_{-\infty} = \ldots \quad u|_{\infty} = \ldots \end{cases}$

Monotone profiles

$$0 = \frac{u'^2}{2}\Big|_{-\infty}^{\infty} + c\int_{-\infty}^{\infty} u'^2 dx + \int_{-\infty}^{\infty} f(u) u' dx = \begin{cases} c\int u'^2 + \int_0^1 f\, du \Rightarrow c < 0 \text{ left} \\ c\int u'^2 - \int_0^1 f\, du \Rightarrow c > 0 \text{ right} \end{cases}$$



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Classical mech. systems in 1D

x – position

$\dot{x} = v$ – velocity

$$\boxed{m\ddot{x} = -f(x, \dot{x}, \dots)}$$

$$(1)\ \begin{cases} \dot{x} = v \\ \dot{v} = \underbrace{f(x)}_{\text{potent.}} - \underbrace{c\,v}_{\text{friction}} \end{cases}$$

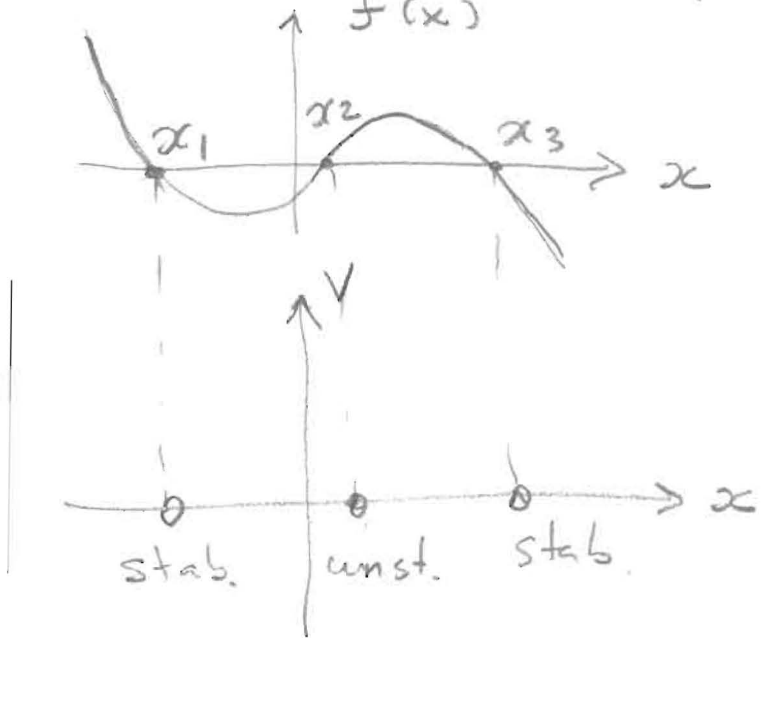
Potential force

$$\boxed{f(x) = -U'(x)}$$

Energy conservation

$E = \dfrac{mv^2}{2} + U(x)$ – const. for any traject. $\begin{matrix} x(t) \\ v(t) \end{matrix}$

Equilibria (1): $\left\{ (x_k, 0) : \boxed{f(x_k) = 0} \right\}$

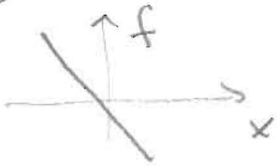
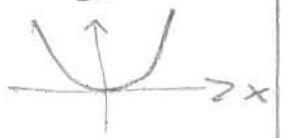
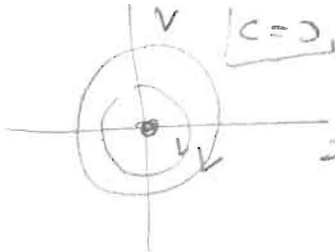
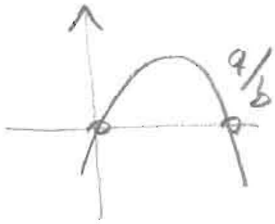
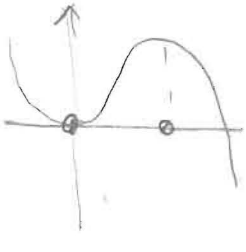
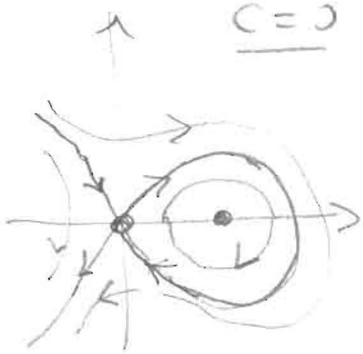
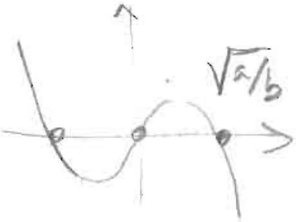
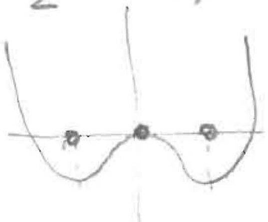
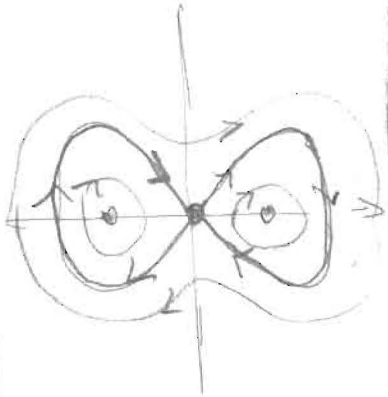


Stability / Jac.

$$J_k = \begin{bmatrix} 0 & 1 \\ f'(x_k) & -c \end{bmatrix}$$

x_1	x_2	x_3
$\begin{bmatrix} 0 & 1 \\ - & -c \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ + & -c \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ - & -c \end{bmatrix}$

Examples:

	$f(x)$	$U(x)$	Ph. plane equil	Jac.
1	$-kx$ 	$\frac{kx^2}{2}$ 	$c=0$ 	$(0,0)$ $\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}$ cent. → sp. sink → sink
2	$ax-bx^2$ 	$\frac{ax^2}{2}-\frac{bx^3}{3}$ 	$c=0$ 	$(0,0)$: $\begin{bmatrix} 0 & 1 \\ a & -c \end{bmatrix}$ saddle $(x_1, 0)$: $\begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix}$ cent. sp. sink sink
3	$ax-bx^3$ 	$\frac{ax^2}{2}-\frac{bx^4}{4}$ 		$(0,0)$: $\begin{bmatrix} 0 & 1 \\ a & -c \end{bmatrix}$ saddle $(\pm x_1, 0)$: $\begin{bmatrix} 0 & 1 \\ \ominus & -c \end{bmatrix}$ cent. sp. sink sink

Friction $(c>0)$:

saddle → saddle

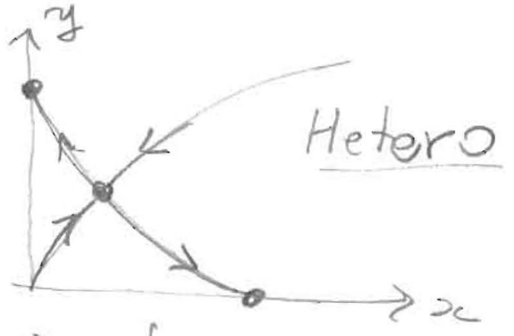
center → sp. sink
→ sink

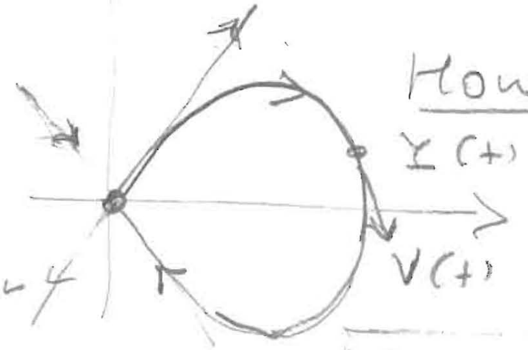
Stable/unstable orbits

(homo- & hetero-clinic)

Each saddle has stable + unstable orbits

(2)-(3)





Exclusive competition

DE: 1) standard orbit: IVP $\boxed{\begin{aligned} \dot{Y} &= F(Y) \\ Y(0) &= Y_0 \end{aligned}}$ V.f.

2) Stable/unst: (n) $\boxed{\ddot{Y} = \underbrace{F'(Y)}_{\text{Jacob.}} \cdot \dot{Y}(t)}$ $\iff$

(2n) $\left\{ \begin{aligned} \dot{Y} &= V \\ \dot{V} &= F'(Y)\cdot V \end{aligned} \right.$ BC: $V(-\infty) = V_-$ - unst. eigen

$V(+\infty) = V_+$ - stab. eigen

Analysis of traveling wave of Fisher-Kolmogorov.

D. Gurarie

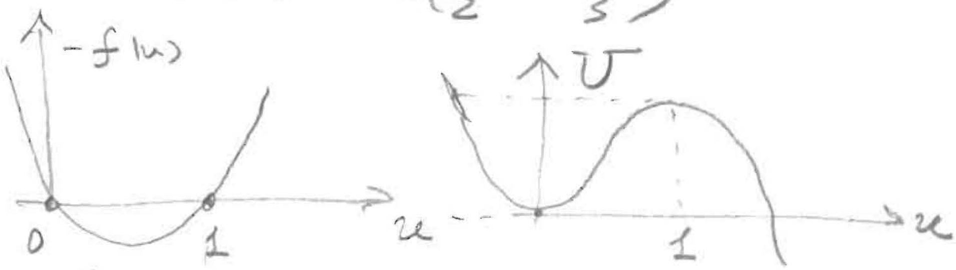
$$(FK)\quad \begin{cases} D u_{xx} + c u_x + f(u) = 0; & f = a u(1-u) \\ u\big|_{-\infty} = 1; \quad u\big|_{+\infty} = 0; \end{cases}$$

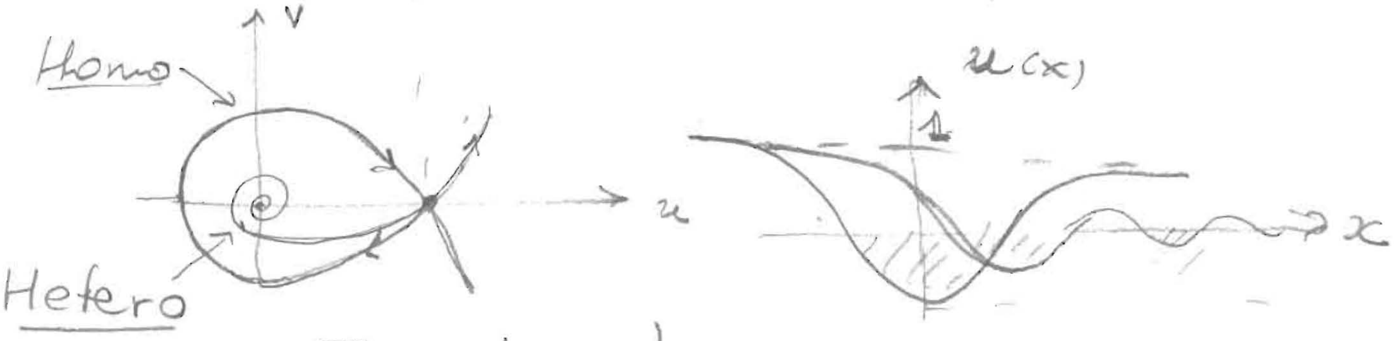
Rescale: $(a, c) \to \left(\frac{a}{D}, \frac{c}{D}\right)$

Write DS: $\boxed{\begin{cases} u' = v \\ v' = -f(u) - cv \end{cases}}$

"Potential & cons. integral $(c=0)$

$$U(u) = a\left(\frac{u^2}{2} - \frac{u^3}{3}\right)$$





For traveling wave $(u(x) \geq 0)$ need bifurcation at $(0,0)$: sp. sink $\to$ sink

$(1,0)$	$(0,0)$
$\begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ a & -c \end{bmatrix}$

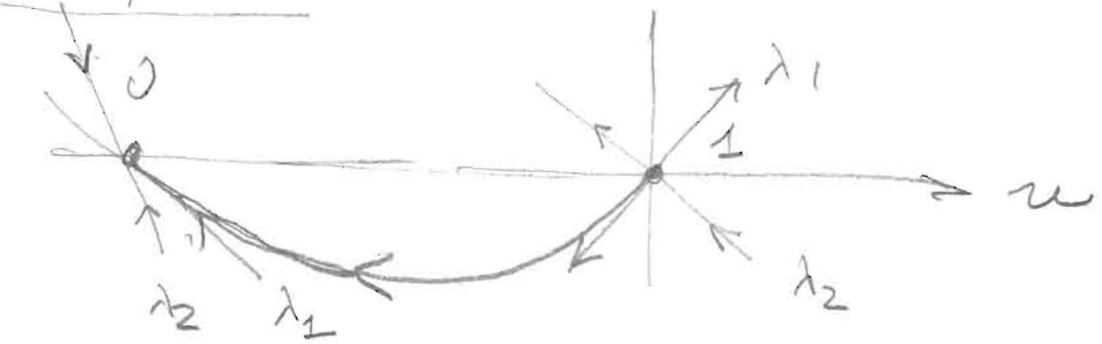
$$\boxed{a = f'(0) = -f'(1)}$$

$$\boxed{\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - a}} \qquad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + a}$$

$0 > \lambda_1 > \lambda_2$

λ_1	λ_2
$\binom{1}{\lambda_1}$	$\binom{1}{\lambda_2}$

λ_1	λ_2
$\binom{1}{\lambda_1}$	$\binom{1}{\lambda_2}$



For all $\boxed{a \le \frac{c^2}{4}} \iff \boxed{c \ge 2\sqrt{a}}$

trav. wave

$$\boxed{\frac{c}{D} \ge 2\sqrt{\frac{a}{D}}} \implies \boxed{c \ge 2\sqrt{aD}}$$

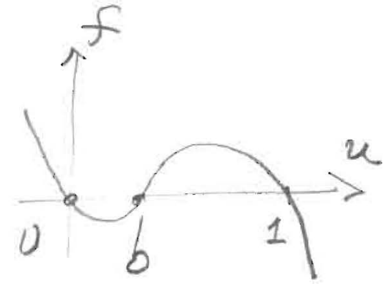
$Du_{xx} + au(1-u) + cu_x$

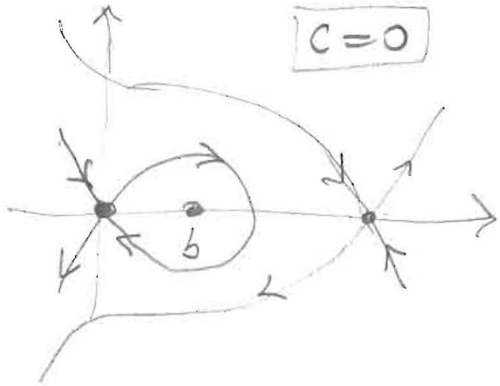
Need sufficiently high $c \ge c_m = 2\sqrt{aD}$

condition for travelling wave

Modified FK

$$u_t = Du_{xx} + \underbrace{a u (u-b)(1-u)}_{f}$$





Condition for monotone wave:

$$c\int_{-\infty}^{\infty} u'^2 \, dx \pm \int_0^1 \underbrace{f(u)\, du}_{} = 0$$

Problem: i) find bifurcation values on (b,c)-parameter space for traveling wave $\boxed{u|_{-\infty}=1; \quad u|_{\infty}=0}$

ii) Demonstrate numerically in phase-plane & solution plots.

Unlike standard FK $f(u) = u(1-u)$ modified $f = u(1-u)(u-b)$ has only one critical speed for each b!

D. GURARIE

Rescaled SIR w. diffusion: spread of rabies by foxes

(*) Healthy foxes stay home
(*) Infected – wonder around (diffuse)
(*) R-group dies (lethal)

$$(1)\begin{cases} S_t = -\beta S I \\ I_t = \beta S I - \gamma I + D I_{xx} \\ R_t = \gamma I \end{cases} \Rightarrow N = \int (S + I + R)\,dx = \text{const}$$

Total

1° Rescale: $\begin{pmatrix} S \\ I \\ R \end{pmatrix} \to \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{N}\begin{pmatrix} S \\ I \\ R \end{pmatrix}$; $t \to \gamma t$, $x \to \sqrt{D/\gamma}\, x$

$$R_0 = \frac{\beta N}{\gamma}$$

D-less: R-D

$$(2)\begin{cases} u_t = -uv \\ v_t = uv - v + v_{xx} \\ w_t = v \end{cases}$$

2° Traveling wave: $\left\{ u(x + ct) \right.$ $\underbrace{}_{= \xi}$ = charact. coord.

$$\Rightarrow (3)\begin{cases} c u' = -R_0 u v \\ v'' - c v' = -(R_0 u - 1) \\ w' = v \end{cases}$$

3rd order

Derivatives: $u' = \frac{du}{d\xi}$; $v' = \ldots$

3° Conserved integral: (2)

$$(4)\begin{cases} \left(\frac{1}{c}V' - V\right)' = -\frac{(R_0 u - 1)}{c} V \\ u' = -\frac{R_0 u V}{c} \\ w' = V \end{cases} \Rightarrow (5)\begin{cases} \frac{d\left(V'/c - V\right)}{du} = 1 - \frac{1}{R_0 u} \\ \frac{dw}{du} = -\frac{1}{R_0 u} \end{cases}$$

eliminate indep. variable ξ

4° Integration of (5)

$$(6)\begin{cases} \frac{V'}{c} - V = \left(u - \frac{1}{R_0}\ln u\right) - A \\ w = -\frac{1}{R_0}\ln u + B \end{cases}$$

constants of integrat

BC:

	$+\infty$	$-\infty$
u	1	u_1
V	0	0
V'	0	0
w	0	w_1

$$\Rightarrow \begin{cases} A = 1 \\ B = 0 \end{cases} \oplus$$

$\oplus$ algebraic eq-n for $\boxed{u_1 + w_1 = 1}$

$$\boxed{1 - w_1 = e^{-R_0 w_1}} \Rightarrow R_0 > 1$$

5° Reduced 2D system:

$$(7)\begin{cases} u' = -\frac{R_0}{c} u V \\ V' = c\left(u - \frac{1}{R_0}\ln u - 1 + V\right) \end{cases}$$

change: $u \to q = -\frac{c}{R_0}\ln u$

Phase-plane analysis

$$(8)\begin{cases} q' = V \\ V' = c\underbrace{\left(e^{-\frac{R_0}{c}q} - 1 - q\right)}_{f(q, R_0)} + cV \end{cases} \Rightarrow$$