2G.

# SIR methodology & extensions

1. Disease age; infectivity strata
2. SIR w. simple demographics
3. Age-structured SIR
   Contract structured SIR
4. Vector-borne: malaria
5. Macro-parasites: mean worm burden (MWB)
6. Evolutionary aspect: mixed strain competition

# 1. Disease age – infectivity strata (3.12)

Pop: $S \xrightarrow{\lambda} I_0 \xrightarrow{\gamma_0} I_1 \xrightarrow{\gamma_1} \dots \to I_n \xrightarrow{\gamma_n} R$

Infectiv: $b_0 \quad b_1 \quad \dots \quad b_n$

FI : $\lambda = \sum b_k I_k$

DS:

$$\begin{cases} \dot S = -\lambda S + \rho R \\ \dot I_0 = \lambda S - \gamma_0 I_0 \\ \vdots \\ \dot I_n = \gamma_{n-1} I_{n-1} - \gamma_n I_n \\ \dot R = \gamma_n I_n - \rho R \end{cases} \quad \overset{?}{\Longrightarrow} \quad \text{Equilibria, BRN}$$

Derivation: $\hat{S} = \xi/\lambda$ ; .. $I_k = \xi/\gamma_k$ ; .. $R = \xi/\rho$

$1^\circ$ $\hat{S} + \sum I_k + R = 1 \Rightarrow \xi = \dfrac{1}{1/\lambda + \underbrace{\sum_k 1/\gamma_k}_{A} + 1/\rho}$

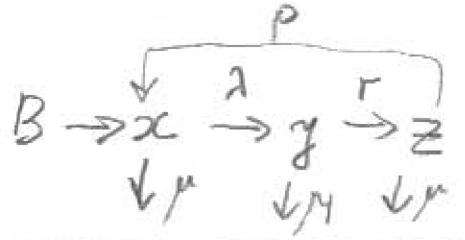
$2^\circ$ $\lambda = \sum b_k I_k = \dfrac{\left(\sum 1/b_k\right)}{1/\lambda + \underbrace{\left(\sum 1/b_k + 1/\rho\right)}_{B}} = \phi(\lambda) = \dfrac{A\lambda}{B\lambda + 1}$


Condition for endemic equil:

$$\boxed{BRN = \phi'(0) = \sum_k b_k/\gamma_k \geq 1}$$

# II. SIR with simple demographics

(birth - death - turnover)




$$\begin{cases} \dot{x} = (B - \mu x) - \lambda x + \rho z \\ \dot{y} = \lambda x - (r+\mu) y \\ \dot{z} = r y - (\rho + \mu) z \end{cases}$$

Total: $h = x + y + z$ $\boxed{\dot{h} = B - \mu h}$ ← Inf. has no effect on $h$

↗ total birth

Stationary demog: $B = b h \iff \boxed{b = \mu}$

↑ pc. birth ↑ pc. death = t/o speed

$\boxed{\lambda = \beta y} \Rightarrow \boxed{R_0 = \frac{\beta}{r + \mu}}$

$x^* = 1/R_0$

$y^* = (1 - 1/R_0) \frac{1/r}{1/r + 1/\rho}$

$z^* = (1 - 1/R_0) \ldots$

* Demographic t/o lowers $R_0$

* Fast t/o $\mu/r > \left(\frac{\beta}{r} - 1\right) \longrightarrow$ eradication

# III. Age structured SIR

| | | |
|---|---|---|
| S | $x_t + x_a = -(\lambda+\mu)x + \ldots$ | **Total** |
| I | $y_t + y_a = \lambda x - (\mu + r) y$ | $n(a,t) = x + y + z$ |
| R | $z_t + z_a = r y - \mu z - \ldots$ | $n_t + n_a = -\mu n$ |
| | | $n\big\|_{a=0} = B + \int_0^\infty b(a)\, n(a,t)\, da$ |

BC: $x\big|_{a=0} = B$; $y\big|_{a=0} = 0$; $z\big|_{a=0} = 0$

Survival (multiplier): $\sigma(a) = e^{-\int_0^a \mu\, d\tau}$

station. pop: $n(a) = n_0\, \sigma(a)$

multiplier solut $\begin{cases} u' = \delta - \gamma u \\ u\big|_0 = u_0 \end{cases} \Rightarrow$ $m(x) = e^{-\int_0^x \gamma\, d\xi}$

$u(x) = u_0\, m(x) + \int_0^x \frac{m(x)}{m(\xi)}\, \delta(\xi)\, d\xi$

| | multipl. | source | solution | $\lambda, r = \text{const}$ |
|---|---|---|---|---|
| $x$ | $e^{-\int(\mu+\lambda)} = \sigma\sigma_\lambda$ | $0$ | $x_0\, \sigma\sigma_\lambda = n(a)\, \sigma_\lambda(a)$ | $\sigma e^{-\lambda a}$ |
| $y$ | $e^{-\int(\mu+r)} = \sigma\sigma_r$ | $\lambda x$ | $\sigma(\sigma_r * \lambda\sigma_\lambda)$ | $\sigma\lambda\left(\frac{e^{-ra} - e^{-\lambda a}}{\lambda - r}\right)$ |
| $z$ | $e^{-\int\mu} = \sigma$ | $r y$ | $\sigma\left(1 * r(\sigma_r * \lambda\sigma_\lambda)\right)$ | $\sigma\frac{\lambda r}{\lambda - r}\left(\frac{1-e^{-ra}}{r} - \frac{1-e^{-\lambda a}}{\lambda}\right)$ |

Const $(\lambda, r, \mu)$: $\sigma = e^{-\mu a}$; $\sigma_\lambda = e^{-\lambda a}$; $\sigma_r = e^{-ra}$; $*$ = convol

## Simple case:

$\boxed{\mu = b}$ - const (age indep.)

FI: $\lambda(a) = \int_0^\infty \boxed{\beta(a,a')}\, y(a')\, da'$

← contact pattern

Assume fixed (age-indep.) contacts:

$$\boxed{\beta(a,a') = \beta} \Rightarrow \boxed{\lambda = \beta Y = \beta \int_0^\infty y\, da}$$

PDE for $(x, y, z)$ turn into ODE for total pop. $X = \int x\, da$; $Y = \int y\, da$; $Z = \int z\, da$

$$\begin{aligned} \dot{X} &= (B - \mu X) - \beta Y X \\ \dot{Y} &= \beta Y X - (r+\mu) Y \\ \dot{Z} &= r Y - (\mu + \ldots) Z \end{aligned}$$

← SIR w. simple demographics

Equilibria: $N = X + Y + Z$ - total

$x(a) = N e^{-(\lambda+\mu)a} = N\sigma e^{-\lambda a}$

$y(a) = N\sigma \dfrac{e^{-\lambda a} - e^{-ra}}{r - \lambda}$

$z(a) = N\sigma \ldots$

$X = \dfrac{\mu}{\lambda+\mu} N$

$Y = \dfrac{\lambda\mu N}{(\lambda+\mu)(r+\mu)}$

$Z = \dfrac{\lambda r N}{(\lambda+\mu)(r+\mu)}$

$$FI = \frac{\beta\mu N}{r+\mu} - \mu = \mu(R_0 - 1)$$ ← "simple $R_0$"

⟨Age⟩ of inf: $\bar{a} = \dfrac{\int a\, \boxed{\lambda x(a)}\, da}{\int \boxed{\lambda x(a)}\, da} = \dfrac{1}{\lambda+\mu} = \dfrac{1}{\mu R_0}$

← incidence

Life expect: $L = \dfrac{\int a\sigma\, da}{\int \sigma\, da} = \dfrac{1}{\mu}$

$\Rightarrow \boxed{R_0 = L/\bar{a}}$ (type II) import. for control.

# Structured transmission

Contact strata:

$X = (x_1 \dots x_n)$
$Y = (y_1 \dots y_n)$
$Z = (z_1 \dots z_n)$

$H = (h_1 \dots h_n)$

Transmission $B = [b_{ij}]$, $b_{ij} = \beta_i \omega_{ij}$ ← cont

Recov: $\hat{R} = diag(R)$; $R = (r_1 \dots r_n)$

Imm. Loss: $\hat{Q} = diag(Q)$; $Q = \dots$

DS:

$$\dot{X} = -(B \cdot Y) X + Q \cdot Z$$
$$\dot{Y} = (B \cdot Y) X - \hat{R} \cdot Y$$
$$\dot{Z} = \hat{R} \cdot Y - \hat{Q} \cdot Z$$

Rescale by $h_i$ ⇒ $x_i + y_i + z_i = 1$

Equilibria: $Z = \frac{\hat{R} \cdot Y}{Q} = \frac{R}{Q} Y$; $X = \frac{R Y}{B \cdot Y} = \dots$

$X + Y + Z = 1 \Rightarrow$ $Y\left(\frac{B \cdot Y}{Q} + 1\right) = (1 - Y) \frac{B \cdot Y}{R}$

$$BRN \longleftrightarrow J\Big|_{(0, \dots 0)} = (\hat{R}^{-1} \cdot B - I)$$

$$\det(\hat{R}^{-1} \cdot B - I) = \begin{cases} > 0 & \text{- endemic.} \\ < 0 & \text{eradic.} \end{cases}$$

$$R_0 \stackrel{?}{=} \lambda_1(\hat{R}^{-1} \cdot B) \geq 1$$

↑ max eigen