SIR with population growth

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SIR population $N = S + I + R$ with birthrate *b*, death rate - μ, transmission - β, recovery - *r*, and disease mortality - δ is given by

$$\dot{S} = bN - \beta SI - \mu S$$
$$\dot{I} = \beta SI - (r + \mu + \delta) I$$
$$\dot{R} = rI - \mu R$$

Problem

1. Show it can be reduced to 2D system for N, I

$$\dot{N} = a(N - \gamma I);$$
$$\dot{I} = \beta(N - N_0 - I) I;$$

with parameters: $a = b - \mu$ - growth rate, $\gamma = \frac{\delta}{a}$, $N_0 = \frac{r + \mu + \delta}{\beta}$.

2. Sketch phase-plane, nullclines, and derive endemic equilibrium: $N^* = \frac{\gamma N_0}{\gamma - 1}; I^* = \frac{N_0}{\gamma - 1}$; provided $\gamma > 1$. What happens to population for $\gamma < 1$, and why $\gamma > 1$ is essential for maintaining equilibrium?
3. Compute Jacobian at (N^*, I^*), $J = \begin{bmatrix} a & -a\gamma \\ \beta I^* & -\beta I^* \end{bmatrix}$; show that

$$\operatorname{tr}(J) = -\left(\frac{r + \mu + a}{\gamma - 1}\right)$$
$$\det(J) = a\beta N_0$$

   and explain stability of equilibrium. Derive formula for BRN $R_0 = \frac{\beta N^*}{r + \mu + \delta} = \frac{\gamma}{\gamma - 1}$

4. Extend the above model and analysis to 2 parasite species (Britton 3.9: Evolutionary aspects) with different $\beta_i; r_i; \delta_i$, and show that least virulent strain (e.g. $\delta_1 < \delta_2$) will win the competition. Find more general condition in terms of $\beta_i; r_i; \delta_i$ for species to succeed (Hint: use $N; I_1; I_2$ formulation, find equilibria and Jacobians). Demonstrate numerically successful invasion of the benign strain.