Review of basic ODE methods

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1. First order DE.

A continuous growth model for a single population $y(t)$ obeys a DE:

$$\frac{dy}{dt}=f(y,t\ldots)=\begin{cases} ry & \text{- linear growth} \\ r(1-y/N)y & \text{-logistic} \\ \ldots & \text{-other} \end{cases} \qquad (1)$$

Coefficient $r>0$, or $r(1-y/N)$ is (density dependent) growth rate, N (for logistic equation) is *carrying capacity*, that constrains growth rate as y approaches N and turns it to negative rate (decay) as y surpasses N.

1) Method of separation for autonomous equations.

The basic method for *autonomous* equation (1) , i.e. $f=f(y)$ is *separation*

$$\frac{dy}{dt}=f(y)\Rightarrow \boxed{F(y)=\int\frac{dy}{f(y)}=C+t},\ C\text{-constant of integration,} \qquad (2)$$

or $\int_{y_0}^{y}\frac{dy}{f(y)}=t-t_0$, for the initial value problem (IVP), $y(t_0)=y_0$. Equation (2) gives implicit solution: $F(y)=t+C$, of DE $y'=f(y)$. Solving it for y we get explicit solution $y(t,C)$.

We list a few specific examples:

Model	DE	Integration	Implicit solution	Explicit solution	IVP
Linear Growth /decay	$y'=ry$	$\int\frac{dy}{y}=rt+c$	$\ln y=c+rt$	$y=Ce^{rt}$	$y=y_0e^{rt}$
Growth with source	$y'=ry+b$	$\int\frac{dy}{y+b/r}=rt+c$	$\ln(y+b/r)$ $=c+rt$	$y=-\frac{b}{r}+Ce^{rt}$	$y=-\frac{b}{r}+\left(y_0+\frac{b}{r}\right)e^{rt}$
Logistic	$y'=r(1-y/N)y$	$\int\frac{dy}{y(1-y/N)}$ $=c+rt$	$\ln\frac{y}{N-y}$ $=c+rt$	$y=\frac{N}{1+Ce^{-rt}}$	$y=\frac{Ny_0}{y_0+(N-y)e^{-rt}}$

2) Linear equations and multipliers

A linear equation: $y' = ay + b$ with constant, or variable coefficients $a = a(t), b = b(t)$ - source/harvest, is solved by multiplier: $\phi(t) = e^{\int_{t_0}^{t} a dt}$. So

$$y(t) = \phi(t)\left[C + \int_{t_0}^{t} \frac{b(s)}{\phi(s)} ds\right], \text{ or } y(t) = \phi(t)\left[y_0 + \int_{t_0}^{t} \frac{b(s)}{\phi(s)} ds\right] \text{ (IVP)}$$

In particular, constant growth rate a, gives exponential $\phi = e^{at}$, and convolution integrals[1]

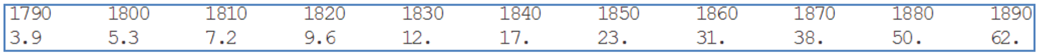
$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-s)} b(s) ds$$

Such system maintains "long term memory" $b(s), (0 < s < t)$ of the source.

Similar method applies to linear differential systems (DS): $Y' = AY + B$, where vector function $Y(t)$ represents the state of a system (e.g. interacting populations), matrix (or matrix function) A is its "generalized growth rate", and $B = B(t)$ - source term. Applications

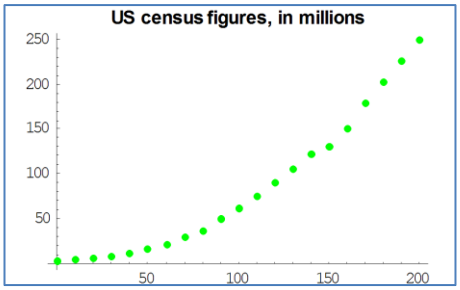
3) Data and model calibration: logistic population growth.

US population census data for years 1790-1990



1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890
3.9	5.3	7.2	9.6	12.	17.	23.	31.	38.	50.	62.

1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
75.	91.	105.	122.	131.	151.	179.	203.	226.	249.



We propose to model it by ***logistic growth***, and ***estimate parameters***: r, N. For 2 parameters, one needs at least 3 data points $y_0 < y_1 < y_2$, at time steps $\Delta t = t_1 - t_0 = t_2 - t_1$, plus analytic solution:

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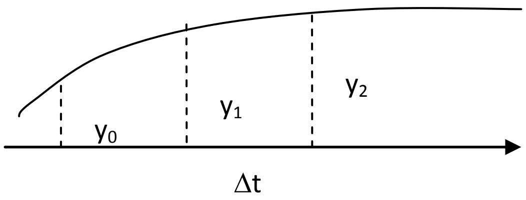
[1] Another method to get such solutions of linear IVP problems (equations, or system) with constant a is Laplace transform



$$y(\Delta t) = \frac{N y_0}{y_0 + (N - y_0) e^{-r\Delta t}} .$$

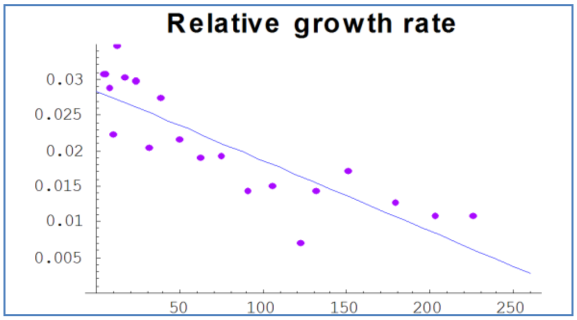
One can solve the resulting algebraic system for r, N

$$x = e^{-r\Delta t} = \frac{1/y_1 - 1/y_2}{1/y_0 - 1/y_1};$$

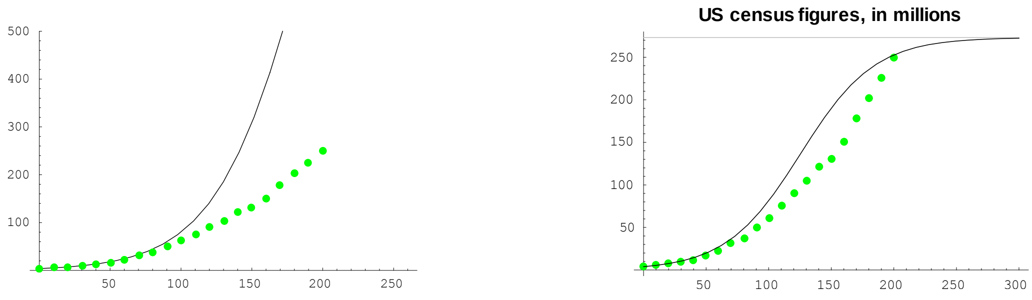
$$N = \frac{1-x}{1/y_0 - x/y_1}$$

Another estimation procedure takes "relative growth rate $r_j = \frac{1}{\Delta t}\ln\left(\frac{y_{j+1}}{y_j}\right)$ vs. population y_j"

and asks for ***best (linear) fit*** of rate function $r(y) = r(1 - y/N)$ larger data



This analysis predicts $r \approx .028, \text{ and } N = 288 \text{ million}$, - overly optimistic.



Logistic prediction: (a) based on 3 data points (left), (b) "best fit" r(y) (right)

2. Equilibria: linearization, stability, bifurcations

1) Linearized stability

An autonomous equation (or system) $y' = f(y)$ has equilibrium solutions: $y = \mathrm{const}$, determined by an algebraic equation (system): $f(y) = 0$. Their stability can be examined by first looking at a ***linearized system***: $u' = mu$, where $m = f'(y_0)$ - derivative (or Jacobian matrix, for DS) at a given equilibrium y_0

$$\begin{cases} m < 0 & \text{stable} \\ m > 0 & \text{unstable} \end{cases}$$

We illustrate it with logistic equation: $f(y) = r(1 - y/N)y$. It has two equilibria: unstable $y = 0,\ f'(0) = r > 0$, and stable carrying capacity, $y = N,\ f'(N) = -r < 0$.

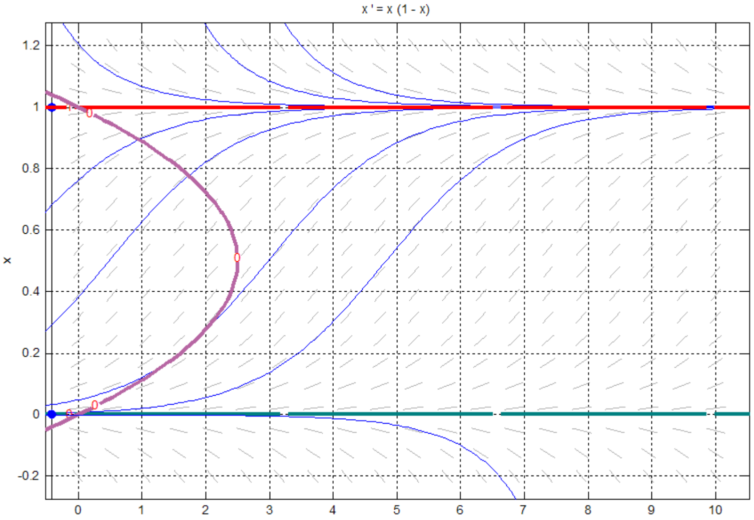


Fig.2: Logistic equilibria and solutions.

Bifurcation analysis.

In many applications differential equations involve some additional parameters, $y' = f(y, b)$, and one would like to know how specific solutions (e.g. equilibria) depend on parameter *b*, or several parameters.

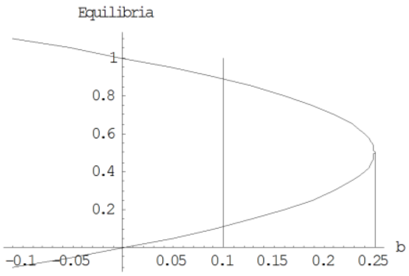
Harvesting Logistic population

A specific example is logistic growth with harvest:

$$y' = r\left(1 - y/N\right)y - b = f\left(y\right) - b \tag{3}$$

where *b* - harvesting rate. While small changes in *b* can change equilibria, they don't change the overall *qualitative* view (*phase-space*) of the system. *Bifurcation* refers to such qualitative changes.

Thus system (3) has a unstable/stable pair $0 < y_1 < y_2 < N$, for all values $0 < b < f_{\max} = rN/2$ The latter marks *bifurcation* value of parameter: $b^* = f_{\max}$, whereby a stable/unstable pair collides and disappears (so called *saddle-node bifurcation*). The bifurcation are often shown by diagrams in the parameter space, e.g. {"b", "equilibria"}



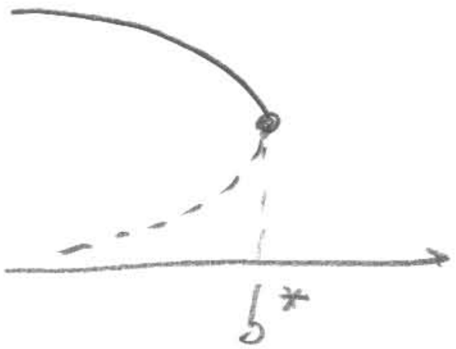
For harvested population, bifurcation has an obvious meaning: harvest rate *b* below the maximal reproduction rate $f_{\max}$ allows to sustain a certain (equilibrium) level of population, whereas $b \geq f_{\max}$ has no equilibria (strictly negative growth) and will wipe out population in finite time.

Basic patterns

Basic patterns:

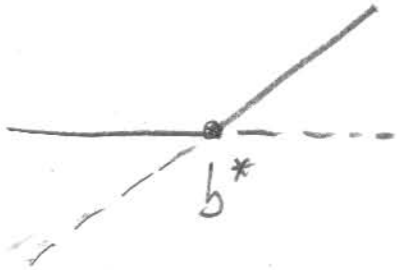
— stable
– – – unstable

Saddle-node



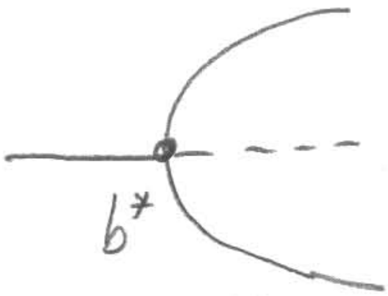
$f = y(1-y) - b$

Transcritical



$f = y(b-y)$

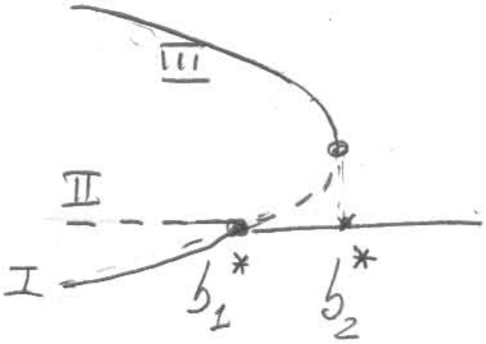
Pitchfork



$f = y(b-y^2)$

Modified logistic:

$f = y^2(1-y) - by$



$b_1^* = 0$ - transcritical

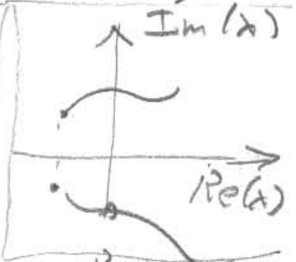
$b_2^* = \frac{1}{4}$ - Saddle-node

Hopf bifurcations

"Equilibria" ↔ "limit cycles"

Complex eigenvalues of Jacobian matrix $J(b)=\left[\frac{\partial f_i}{\partial x_j}\right]$ of v.f. $F=\begin{pmatrix} \vdots \\ f_i(x_1 \dots x_n) \\ \vdots \end{pmatrix}$

crossing imaginary line: $Re(\lambda_k)$



Linearized solutions: $\lambda=\alpha\pm i\beta$

$Ce^{\lambda t}=e^{\alpha t}(\cos\beta t+i\sin\beta t)$ → stable: $Re\,\lambda<0$

→ unstable: $Re\,\lambda>0$

Examples: complex variable $z=x+iy$ in $\mathbb{C}=\mathbb{R}^2$

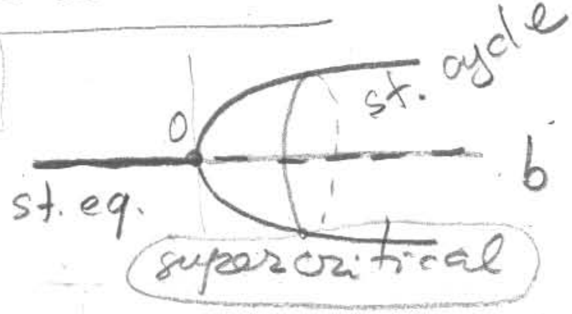
real f

DS: $\dot{z}=\left[i\omega+f(|z|^2)\right]z$

Polar: $z=re^{i\theta} \Longrightarrow \begin{cases}\dot{r}=f(r^2)r \\ \dot{\theta}=\omega\end{cases}$

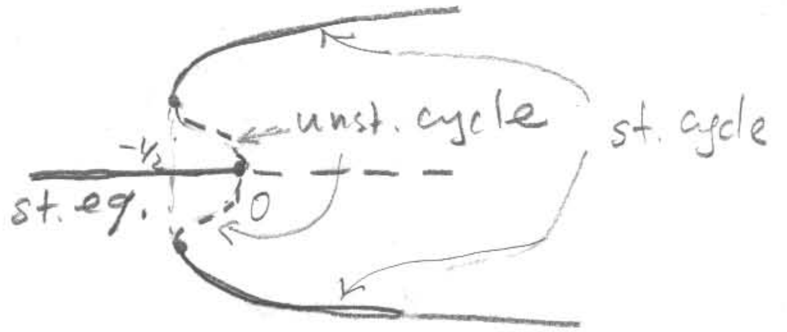
1° $f=(b-r^2)\Rightarrow \boxed{\dot{r}=(b-r^2)r}$

$J=\begin{bmatrix} b & -\omega \\ \omega & b\end{bmatrix}$; $\boxed{\lambda=b\pm i\omega}$



2° $f=(b+r^2-r^4)$;

Subritical



Spruce budworm

is one of the most severe forest defoliators in North America. Its life cycle and outbreaks are documented in the enclosed web-links. We shall consider a simple logistic model of budworm population density $y(t)$ with carrying capacity N (due to environmental constrains), and an additional removal (predation) term $P(y)$ (by birds),

$$y' = r_B(1 - y/N)y - P(y) \tag{4}$$

The form of P depends on predation patterns, a natural choice is satiated predation $P = \frac{By^2}{A^2 + y^2}$ - a sigmoid function with threshold level A, and B - proportional to predator density. We are interested in the effect of predation on sustained budworm infestation level.

This problem has now two parameters, so one has to study the A, B - parameter space.

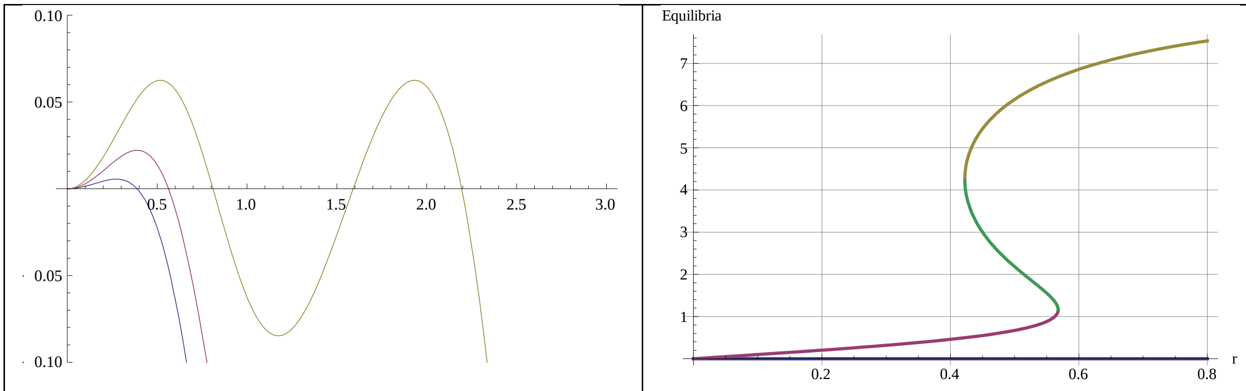
It can be rescaled to a dimensionless form by changing: $y \to u = y/A$, along with time and other parameters:

$u = y/A$	- Budworm population
$\tau = tB/A$	- time
$q = N/A;$	- C.C.
$r = \frac{r_B A}{B}$	- max reproduction rate

The resulting system depends on (r, q) ,

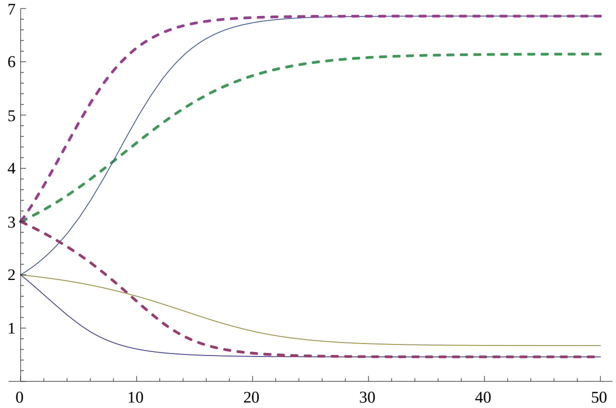
$$u' = r(1 - u/q)u - \frac{u^2}{1 + u^2} = f(u, r, q) \tag{5}$$

where r is inverse proportional to predation rate (bird population) B. Fig.1 shows (A) typical functions $f(u, r, q)$ with 2-4 equilibria, (B) its bifurcation diagram at fixed q=9, (C) dynamic patterns u(t) that exhibit transition from "low" to "high" stable equilibria (infestation outbreak) at certain values r (decreased predation). More detailed analysis of equilibria and bifurcations of (5) is given in *Mathematica* notebook



(A) functions f(u,r,q) with fixed q=8 and 3 values r={.15,.3,.5}

(B) bifurcation diagram of f(u,r,9): stable branches (purple, olive) and unstable (blue, green)



(C) dynamic patterns u(t) with 2 different initial states u_0 ={2,3}, and r=(.4,.5,.6)

Figure 1

An important transition factor for outbreak is changing predation level B. To find its effect one use a different scaling, based on (fixed) carrying capacity and basic reproduction r_B, i.e.

$y \to y/N;\ t \to r_B t$.

3. *Problems*

1. Rescale (4) to nondimensional form

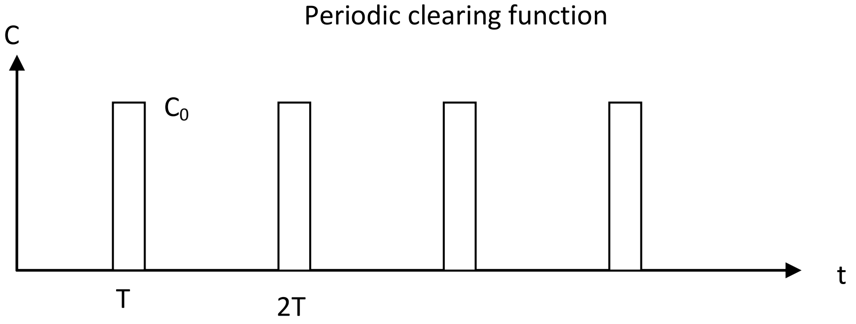
$$\frac{du}{dt} = (1-u)u - \frac{bu^2}{a^2+u^2} \tag{6}$$

and derive parameters a,b. Plot bifurcation diagram for b (with fixed a) similar to Fig.1(B), and compute the a,b-plane diagram. Study numerically the effect of slowly dropping predation level $b(t)=.3e^{-\varepsilon t};\ (\varepsilon=.02)$, and estimate the time of outbreak.

2. Consider periodic pest-control for (6) with slowly decaying predation b of problem 1 , given by step- function $C(t)$, $\frac{du}{dt}=(1-u)u-\frac{bu^2}{a^2+u^2}-C(t)u$

Each step (time intervals T) has short duration $\Delta t << T$, and high removal rate C_0. So each treatment kills fraction $f=1-e^{-C_0\Delta t}$ of pest. Assume 90% treatment efficiency (f=.1), take subcritical b (high infestation level) and determine (numerically) a suitable control T that could prevent outbreaks. Hint: show the long term effect of periodic treatment is equivalent to a fixed (effective) decay rate for $C_{eff}=\frac{C_0\Delta t}{T}=\frac{\ln(1-f)}{T}$. Compare two models: periodically treated (6) and "γ-decay model". Use bifurcation diagram and values of **problem 1**.

Periodic clearing function



3. Second order equations (solves by separation).

Method of separation extends to other models, and higher order equations, for instance, mechanical systems with *potential force*: $\boxed{m\ddot{x} = f(x) = -U'(x)}$, $U(x)$ - potential energy. Here $x(t)$ -(particle) position function, f -force. They include among other,

i) linear and non-linear oscillators: $U = \frac{kx^2}{2}$ (k - spring constant), $f = -kx$, or

ii) $U = \frac{ax^4}{4} - \frac{bx^2}{2}$ (Duffing "double-well") with force $f = -ax^3 + bx$,

iii) pendulum: $l\ddot{\theta} = -g\sin\theta,\ U = -g\cos\theta$ (l - length, g - gravity constant)

All those systems have *conserved integral* (*energy*) $E = \frac{m\dot{x}^2}{2} + U(x)$ - const, which allows to reduce 2-nd order equation to a first order (autonomous) family of equations:

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

solved by separation

$$\int dx \Big/ \sqrt{\frac{2}{m}[E - U(x)]} = t\,,$$

In particular period of an oscillatory solution,

$$T = \sqrt{\frac{m}{2}}\int_{x_-}^{x_+} \frac{dx}{\sqrt{[E - U(x)]}}$$

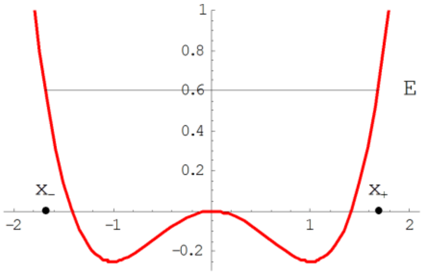


Fig.1: Duffing oscillator potential

Special cases include:

1. Linear oscillator: $U = \frac{kx^2}{2}$ yields trig. (exponential) solutions

   $x(t) = A\cos(\omega t - \phi)$, A - amplitude, $\omega = \sqrt{k/m}$ - frequency, ϕ -phase shift

2. Pendulum: $U = -\frac{g}{l}\cos\theta$, yields periodic Jacobi amplitude solution (same as Duffing oscillator)

   For further details see Mathematica notebook.

Remark: Reduction (to first order) method works only for special second order equations. More general and useful approach is to convert higher order equations to differential systems (DS) and use phase space analysis (later).