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Population dynamics:

(discrete age + continuous time)

Demographics:

Age bins (duration)	L_0	L_1	...	L_m
Age-spec. mortality	μ_0	μ_1		
Age-spec. birth rate		b_1	---	b_m

Maturation rates: $\alpha_k = 1/L_k$; $k=0,1,\ldots$

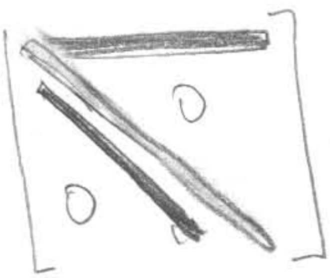
Age-specific fecundity: $f_k = b_k L_k = \frac{b_k}{\alpha_k}$ – (average # newborn)

State: $\underline{Y(t)} = (y_0, \ldots y_m)$

DS:
$$\left\{\begin{aligned} \dot{y}_0 &= -(\mu_0+\boxed{\alpha_0})y_0 + \sum_{k=1}^{m} b_k y_k \\ \dot{y}_1 &= \alpha_0 y_0 - (\alpha_1+\mu_1) y_1 \\ &- - - - - - \\ \dot{y}_m &= \alpha_{m-1} y_{m-1} - \alpha_m y_m \end{aligned}\right.\; ; \quad \boxed{\dot{\underline{Y}} = A \cdot \underline{Y}}$$

Continuous (Leslie-type)

$$A = \begin{bmatrix} -(\alpha_0+\mu_0) & b_1 & b_2 & \ldots & b_m \\ \alpha_0 & -(\alpha_1+\mu_1) & 0 & \ldots & \\ & \alpha_1 & -(\cdot) & 0 & - \\ & & & \alpha_{m-1} & -\alpha_m \end{bmatrix},$$



2)

Eigenvalue analysis (growth-decay)

$$\boxed{A \cdot X = \lambda X} \rightarrow X = \begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix}$$

$$\Rightarrow \quad x_k = \left(\frac{\alpha_{k-1}}{\lambda + \alpha_k + \mu_k}\right) x_{k-1}; \quad k \geq 2$$

$$x_m = \frac{\alpha_{m-1}}{\lambda + \alpha_m} \, x_{m-1}$$

Introduce notation

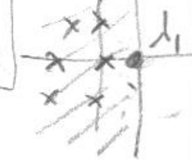
$$\phi_k = \lambda + (\alpha_k + \mu_k)$$

Eigenvector: $x_k = \dfrac{\alpha_0 \ldots \alpha_{k-1}}{\phi_1 \ldots \phi_k} \boxed{x_0}$ & $\phi_0 x_0 = \sum_1^m b\, x_k$

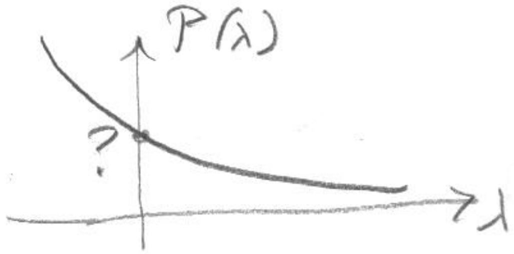
Charact. eq-n:

$$\boxed{P(\lambda) = \frac{\alpha_0}{\phi_0}\left(\frac{b_1}{\phi_1} + \frac{\alpha_1}{\phi_1}\frac{b_2}{\phi_2} + \ldots + \frac{\alpha_1 \ldots \alpha_{m-1} b_m}{\phi_1 \ldots \phi_{m-1}\phi_m}\right) = 1}$$

Thm: Matrix A has largest real eigen λ_1 with $X_1 \geq 0$; all other $\boxed{Re(\lambda_k) \leq \lambda_1}$



Stability analysis:



$$BRN = P(0) = \begin{cases} > 1 \Rightarrow \lambda_1 > 0 \text{ - growth} \\ = 1 \Rightarrow \lambda_1 = 0 \text{ - sustain} \\ < 1 \Rightarrow \lambda_1 < 0 \text{ - decay} \end{cases}$$

$$X(t) \approx c_1 e^{\lambda_1 t} X_1 + \ldots$$

Q: which fecundity levels $\{f_i\}$ would sustain population: $\lambda_1 \geq 0$ or $\boxed{P(0) \geq 1}$

Notation: $\beta_K = \left.\frac{\alpha_K}{\phi_K}\right|_{\lambda=0} = \frac{1}{1+\mu_K/\alpha_K}$; $b_k = \alpha_k f_k$

$$P(0) = \beta_0 \left(\beta_1 f_1 + \beta_1 \beta_2 f_2 + \dots \right) \overset{?}{=} 1$$

Examples:

1) Child-adult: $(m=1)$ $\frac{d}{dt}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} -(\alpha_0+\mu_0) & b_1 \\ \alpha_0 & -\alpha_1 \end{bmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

$(\beta_1 = 1) \Rightarrow \boxed{f_1 = 1/\beta_0 = 1 + \frac{\mu_0}{\alpha_0}}$ ← replacement fertility

(#children / family)

2) $m=2$: $(y_0 \; y_1 \; y_3) \rightarrow (\beta_2 = 1)$

$$\beta_0 \beta_1 (f_1 + f_2) = 1$$

Total: $\boxed{f = (f_1 + f_2) = \frac{1}{\beta_0 \beta_1} = \left(1 + \frac{\mu_0}{\alpha_0}\right)\left(1 + \frac{\mu_1}{\alpha_1}\right)}$