

Another observation about operator compressions

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joint work with Mark Meckes

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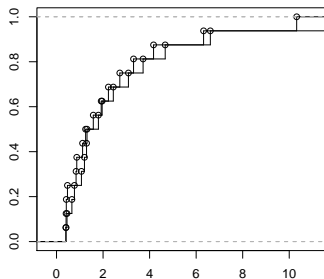
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For M given, let A be chosen uniformly at random from all $k \times k$ principal submatrices. Let F_A denote the empirical distribution function of A ; that is,

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Let $F(x) := \mathbb{E}F_A(x)$. Then for $r > 0$,

$$\mathbb{P}[\|F_A - F\|_\infty \geq k^{-1/2} + r] \leq 12\sqrt{k}e^{-r\sqrt{k/8}}$$

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and

$$\mathbb{E}\|F_A - F\|_\infty \leq \frac{13 + \sqrt{8} \log(k)}{\sqrt{k}}.$$

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1. Functions of finite state space Markov chains are concentrated at their means with respect to the stationary distribution of the chain, with bounds in terms of the spectral gap of the chain.
2. The transposition random walk on S_n has uniform measure as stationary distribution.
3. Good bounds (due to Diaconis and Shahshahani) on the spectral gap of the transposition random walk are available.
4. It's not too hard to get from concentration of $F_A(x)$ near $F(x)$ to concentration of $\|F_A - F\|_\infty$.

Away from matrices: the coordinate-free viewpoint

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$$T_E := \pi_E T|_E = \pi_E T \pi_E^*.$$

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In particular, if $\mathcal{H} = \mathbb{R}^n$ and E is a coordinate subspace, then T_E is a principal submatrix of the matrix of T .

The spectral distribution of T_E is defined to be the measure

$$\mu_E := \frac{1}{k} \sum_{j=1}^k \delta_{\lambda_j(T_A)},$$

where $\lambda_1(T_A) \geq \dots \geq \lambda_k(T_A)$ are the eigenvalues of T_A .

Our observation

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For T a given self-adjoint operator on an n -dimensional Hilbert space \mathcal{H} and $1 \leq k \leq n$, most compressions of T to k -dimensional subspaces have spectral distributions which are about the same.

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Let E be a random subspace of \mathcal{H} distributed according to the rotation-invariant probability measure on the Grassmannian, let μ_E be the spectral measure of T_E and $\mu := \mathbb{E}\mu_E$.

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$$\rho(T) := \frac{1}{2} [\lambda_1(T) - \lambda_n(T)] \quad \sigma_k(T) := \inf_{\lambda} \sqrt{\sum_{i=1}^k s_i^2(T - \lambda I)}.$$

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Then

$$\mathbb{E}d_1(\mu_E, \mu) \leq A \frac{\sigma_k(T)^{4/7} \rho(T)^{3/7}}{(kn)^{2/7}},$$

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and

$$\mathbb{P} \left[d_1(\mu_E, \mu) \geq A \frac{\sigma_k(T)^{4/7} \rho(T)^{3/7}}{(kn)^{2/7}} + t \right] \leq B \exp \left[-C \frac{knt^2}{\sigma_k^2(T)} \right].$$

A note on distance

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We use the Kantorovich-Rubenstein distance

$$\begin{aligned}d_1(\mu, \nu) &= \inf_{\pi} \int_{\mathbb{R} \times \mathbb{R}} |x - y| d\pi(x, y) \\ &= \sup_f \left| \int f d\mu - \int f d\nu \right| \\ &= \|F_{\mu} - F_{\nu}\|_{L_1(\mathbb{R})},\end{aligned}$$

where π varies over probability measures on $\mathbb{R} \times \mathbb{R}$ with margins μ and ν , and f varies over functions on \mathbb{R} with $\|f'\|_{\infty} \leq 1$.

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where π varies over probability measures on $\mathbb{R} \times \mathbb{R}$ with margins μ and ν , and f varies over functions on \mathbb{R} with $\|f'\|_{\infty} \leq 1$.

This distance is not directly comparable to the Kolmogorov distance $\|F_{\mu} - F_{\nu}\|_{\infty}$ in general, although some comparison can be made here due to the finite support of the measures in question.

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Recall that one can define a metric $d(E, F)$ on $\mathfrak{G}_k(\mathcal{H})$ by

$$d(E, F) := \inf \sqrt{\sum_{i=1}^k \|e_i - f_i\|^2},$$

where the infimum is over orthonormal bases $\{e_i\}_{i=1}^k$ and $\{f_i\}_{i=1}^k$ of E and F , respectively.

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One has concentration about a fixed value for functions on $\mathfrak{G}_k(\mathcal{H})$ which are Lipschitz with respect to the distance $d(\cdot, \cdot)$.

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Theorem (Gromov-Milman)

Let $f : \mathfrak{G}_k(\mathcal{H}) \rightarrow \mathbb{R}$ be 1-Lipschitz with respect to $d(\cdot, \cdot)$, and let E be distributed according to the rotation-invariant probability measure on $\mathfrak{G}_k(\mathcal{H})$. Then there are absolute constants C, c such that

$$\mathbb{P} [|f(E) - \mathbb{E}f(E)| \geq t] \leq Ce^{-cnt^2}.$$

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We want to apply this theorem to the function $f(E) := d_1(\mu_E, \mu)$, where d_1 is the Kantorovich-Rubenstein distance, μ_E is the spectral distribution of the compression of T to E , and $\mu = \mathbb{E}\mu_E$.

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Then

$$d_1(\mu_E, \mu_F) \leq \frac{1}{k} \sum_{i=1}^k |\lambda_i(T_E) - \lambda_i(T_F)| \leq \sqrt{\frac{1}{k} \sum_{i=1}^k |\lambda_i(T_E) - \lambda_i(T_F)|^2}.$$

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Lidskii's theorem gives that for $k \times k$ Hermitian matrices A, B ,

$$\sqrt{\sum_{i=1}^k |\lambda_i(A) - \lambda_i(B)|^2} \leq \|A - B\|_{HS} = \sqrt{\sum_{i,j=1}^k |a_{ij} - b_{ij}|^2}.$$

It follows that

$$\begin{aligned}d_1(\mu_E, \mu_F) &\leq \sqrt{\frac{1}{k} \sum_{i,j=1}^k |\langle T(\mathbf{e}_j), \mathbf{e}_i \rangle - \langle T(\mathbf{f}_j), \mathbf{f}_i \rangle|^2} \\&\leq \sqrt{\frac{1}{k} \sum_{i,j=1}^k [\|T(\mathbf{e}_j)\| \|\mathbf{e}_i - \mathbf{f}_i\| + \|T(\mathbf{f}_j)\| \|\mathbf{e}_j - \mathbf{f}_j\|]^2} \\&\leq \frac{d(E, F)}{\sqrt{k}} \left(\sqrt{\sum_{j=1}^k \|T(\mathbf{e}_j)\|^2} + \sqrt{\sum_{i=1}^k \|T(\mathbf{f}_i)\|^2} \right) \\&\leq \frac{2d(E, F)}{\sqrt{k}} \|T\|_{(k),2}.\end{aligned}$$

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Since $d_1(\mu_E, \mu_F)$ is invariant under addition of a real scalar matrix to T , one can optimize over such additions to get σ_k in place of $\|T\|_{(k),2}$.

This bound gives the Lipschitz constant of $f(E) = d_1(\mu_E, \mu)$,
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Recall that $d_1(\mu_E, \mu) = \sup_f \left| \int f d\mu_E - \int f d\mu \right|$; we need to bound the expected maximum of a stochastic process indexed by $\{f : \|f'\|_\infty \leq 1\}$.

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Let $\{X_t\}_{t \in T}$ be a stochastic process indexed by a metric space T with distance d . Suppose that there is a constant c such that X_t satisfies the increment condition

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Then there is a constant C such that

$$\mathbb{E} \sup_{t \in T} X_t \leq C \int_0^\infty \sqrt{\log N(T, d, \epsilon)} d\epsilon,$$

where $N(T, d, \epsilon)$ is the ϵ -covering number of T with respect to the distance d .

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- ▶ The measure-concentration result on $\mathfrak{G}_k(\mathcal{F})$ thus gives that

$$\mathbb{P}[|X_f - X_g| > u] \leq \mathbb{P}[|X_{f-g}| \geq u] \leq C \exp \left[-c \frac{nk u^2}{\sigma_k^2 |f-g|_L} \right],$$

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⇒ The process satisfies the sub-Gaussian increment condition with respect to $\|\cdot\|' := \frac{\sigma_k}{\sqrt{kn}} \|\cdot\|_{C^1}$.

Bad News: The covering number of $\{f : \|f'\|_\infty \leq 1\}$ with respect to $\|\cdot\|'$ is infinite. In fact, it suffices to consider $\{f : \|f\|_{C^1} \leq 1 + 2\rho\}$, but that still has infinite covering number.

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We get the estimate we want by smoothing functions in $\{f : \|f\|_{C^1} \leq 1 + 2\rho\}$: let $f_t = f * \varphi_t$, with $\varphi_t(x) := \frac{1}{t} \varphi\left(\frac{x}{t}\right)$ and φ a smooth probability density with finite first moment and $\varphi' \in L_1(\mathbb{R})$.

Keeping track of how $\|f_t\|_{C^2}$ depends on $\|f\|_{C^2}$ and t , and optimizing over t , yields

$$\mathbb{E}d_1(\mu_E, \mu) \lesssim \frac{\sqrt{\sigma_k}(1+\rho)^{1/4}}{(kn)^{1/4}} + \frac{\sigma_k(1+\rho)^{3/2}}{\sqrt{kn}}.$$

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Exploiting the fact that $d_1(\mu_E, \mu)$, σ_k , and ρ are all homogeneous of order 1 when T is multiplied by a positive constant, one can optimize over that factor to obtain the bound in the statement of the theorem:

$$\mathbb{E}d_1(\mu_E, \mu) \lesssim \frac{\sigma_k^{4/7} \rho^{3/7}}{(kn)^{2/7}}.$$

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- ▶ The random subspace onto which the operator is compressed is distributed differently: in C-L, E is uniformly chosen from the coordinate-subspaces of dimension k , whereas for us, E is uniformly chosen from *all* subspaces of dimension k . As such, quantitative comparison is necessarily rough.
- ▶ The C-L result is a concentration result for $\|F_{\mu_E} - F_{\mu}\|_{\infty}$ and a bound on its mean, whereas our result is a concentration result for $d_1(\mu_E, \mu)$ and a bound on its mean.

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- ▶ The C-L result is a concentration result for $\|F_{\mu_E} - F_{\mu}\|_{\infty}$ and a bound on its mean, whereas our result is a concentration result for $d_1(\mu_E, \mu)$ and a bound on its mean. Normally, the two metrics are not comparable; however, in this case the measures in question are supported on $[\lambda_n(T), \lambda_1(T)]$, which implies that $d_1(\mu_E, \mu) \leq 2\rho\|F_{\mu_E} - F_{\mu}\|_{\infty}$. This allows some quantitative comparison.

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- ▶ In particular, our result shows that the fluctuations of $d_1(\mu_E, \mu)$ about its mean are of order roughly $(kn)^{-1/2}\sigma_k(T) \leq n^{-1/2}\rho(T)$. Making use of the bound $d_1(\mu_E, \mu) \leq 2\rho\|F_{\mu_E} - F_{\mu}\|_{\infty}$ together with the C-L result gives a bound on the fluctuations of d_1 about its mean of order $k^{-1/2}\rho(T)$.

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- ▶ Comparing the bounds on the expected distances is harder; the upshot is that ours seems better in some regimes and theirs seems better in others.

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- ▶ Observe that if T is a scalar matrix, then μ_E is a point mass at μ .
- ▶ Our results see (through their dependence on ρ and σ_k), that if T is “nearly scalar”, then the concentration is stronger and the expected distance is smaller than in the general case.

Thank you.