Homework 6

1. Suppose that n boys and n girls are assigned colors from a collection of c possible colors, independently and uniformly. By proving a Poisson approximation theorem, determine how large n must be so that the probability that some boy and some girl have the same color is at least $\frac{1}{2}$, if c is large.

2. Let $a = (a_1, \ldots, a_n)$ be such that $\sum_{i=1}^n a_i^2 = 1$. Let $\{X_i, \ldots, X_n\}$ be i.i.d. random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Use Stein's method to estimate the distance from $W := \sum_{i=1}^n a_i X_i$ to a standard Gaussian distribution. Give examples of choices of a for which W is approximately Gaussian and choices for which a central limit theorem does not (and should not) follow from your bound.

3. *Approximate matching.* Let $\sigma \in S_n$ be a random permutation, and let $k \in \{1, \ldots, n\}$. Let $W = \sum_{i=1}^n \mathbb{1}(|\sigma(i) - i| \leq k)$, where the circular convention is used, so that e.g. $|n-1| = 1$. Use Stein's method to estimate the distance from W to the Poisson distribution with the same mean.