Extra problems for Homework 5

1. Let $d_{TV}(\mu,\nu) := \sup_A \big|\mu(A) - \nu(A)\big|$, where μ and ν are probability measures on a measurable space $(\Omega, \mathcal{F})$ and the supremum is over measurable sets A. Prove:

   (a) The total variation distance is a metric on the space of probability measures on $(\Omega, \mathcal{F})$.

   (b) The definition above is equivalent to
   $$d_{TV}(\mu,\nu) := \frac{1}{2} \sup_{f:\Omega\to[-1,1]} \left| \int f d\mu - \int f d\nu \right|,$$
   where the functions f in the supremum are required to be measurable.

   (c) If μ and ν have densities f and g with respect to λ, show that
   $$d_{TV}(\mu,\nu) = \frac{1}{2} \int |f-g| d\lambda.$$

2. Consider an exponential random variable Y_λ with parameter λ, and let $\mathcal{F}_o$ be the set of bounded C^1 funtions f on $\mathbb{R}_+$ with $f(0) = 0$, such that $\lim_{x\to\infty} f(x)e^{-\lambda x} = 0$. Let $\mathcal{X}_o$ be the set of bounded measurable functions on $\mathbb{R}_+$ such that $\mathbb{E} f(Y_\lambda) < \infty$. Define $T_o : \mathcal{F}_o \to \mathcal{X}_o$ by
   $$T_o f(x) = f'(x) - \lambda f(x).$$
   Prove:

   (a) T_o really does map $\mathcal{F}_o$ into $\mathcal{X}_o$.

   (b) $\mathbb{E} T_o f(Y_\lambda) = 0$ for all $f \in \mathcal{F}_o$.

   (c) Define $U_o : \mathcal{X}_o \to \mathcal{F}_o$ by
   $$U_o g(x) := e^{\lambda x} \int_0^x [g(t) - \mathbb{E} g(Y_\lambda)] e^{-\lambda t} dt.$$
   Show that U_o really does map $\mathcal{X}_o$ into $\mathcal{F}_o$ and that $T_o U_o g(x) = g(x) - \mathbb{E} g(Y_\lambda)$.

   (d) Show that if X is a random variable taking values in $\mathbb{R}_+$ such that $\mathbb{E} T_o f(X) = 0$ for all $f \in \mathcal{F}_o$, then X is an exponential random variable with parameter λ.

   That is, T_o is a characterizing operator for $\mathcal{Y}_\lambda$.