

# Homework 10

1. Show that a finite state space Markov chain has the strong Markov property; that is, if  $\{X_n\}_{n=0}^\infty$  is a Markov chain on  $\Omega$  with transition matrix  $P$ , and  $\tau$  is a stopping time with respect to the filtration  $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$ , show that

$$\mathbb{P}_{x_o} [(X_{\tau+1}, \dots, X_{\tau+k}) \in A | X_\tau] = \mathbb{P}_{X_\tau} [(X_1, \dots, X_k) \in A],$$

for any  $x_o \in \Omega$ , and that conditional on  $X_\tau$ ,  $(X_{\tau+1}, \dots, X_{\tau+k})$  is independent of  $(X_1, \dots, X_{\tau-1})$ .

2. (An alternate proof of the existence of stationary distributions). Let  $P$  be the transition matrix of a Markov chain on  $\Omega$ . For each  $n \in \mathbb{N}$ , define

$$\nu_n := \frac{1}{n} [\mu + \mu P + \dots + \mu P^n].$$

- (a) Show that for all  $x \in \Omega$  and  $n \in \mathbb{N}$ ,

$$|\nu_n P(x) - \nu_n(x)| \leq \frac{1}{n}.$$

- (b) Show that there is a subsequence  $\{\nu_{n_k}\}_{k=1}^\infty$  of  $\{\nu_n\}_{n=1}^\infty$  such that  $\lim_{k \rightarrow \infty} \nu_{n_k}(x)$  exists for each  $x \in \Omega$ .

- (c) Let  $\nu(x) = \lim_{k \rightarrow \infty} \nu_{n_k}(x)$ ; show that  $\nu$  is a stationary distribution for  $P$ .

3. Let  $P$  be an irreducible transition matrix of period  $m$ . Show that  $\Omega$  can be partitioned into  $\mathcal{C}_1, \dots, \mathcal{C}_m$  such that  $P(x, y) > 0$  only if there is an  $i$  with  $x \in \mathcal{C}_i$  and  $y \in \mathcal{C}_{i+1}$  (with addition in the subscript being addition mod  $m$ ).
4. Let  $P$  be the transition matrix of a Markov chain on  $\Omega$  and let  $\mu$  and  $\nu$  be probability measures on  $\Omega$ . Show that

$$d_{TV}(\mu P, \nu P) \leq d_{TV}(\mu, \nu).$$

Use this to show that another application of  $P$  can only move the chain closer to stationarity.