Math 423 Homework 6

1. Let $f \in L^1(m)$ and let $F(x) := \int_{(-\infty,x]} f dm$. Show that F is continuous.

2. Show that $\int_0^\infty x^n e^{-x} dx = n!$ by differentiating (with justification) $\int_0^\infty e^{-tx} dx = \frac{1}{t}$. Similarly, show that $\int_{-\infty}^\infty x^{2n} e^{-x^2} dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ by differentiating the equation $\int_{-\infty}^\infty e^{tx^2} dx = \sqrt{\frac{\pi}{t}}$.

3. Suppose that $\{f_n\} \subseteq L^1$ and that there is a $g \in L^1$ with $|f_n| \leq g$ for all n. Suppose further that $f_n \to f$ in measure. Show that $f_n \to f$ in L^1, and that consequently, $\int f d\mu = \lim_n \int f_n d\mu$.

4. Suppose that $f_n \to f$ in measure.

   (a) If φ is uniformly continuous, then $\varphi \circ f_n \to \varphi \circ f$ in measure.

   (b) If $\mu(X) < \infty$, then $\varphi \circ f_n \to \varphi \circ f$ in measure whenever φ is continuous (i.e., φ does not need to be uniformly continuous in this case).