Math 423 Homework 12

1. Show that in any topological space, a finite union of compact sets is compact, and that if the space is Hausdorff, then an arbitrary intersection of compact sets is compact.

2. Form a topological space X by taking $(-1,1)$, and add to it a point called 0^* (the idea is to split 0 into two parts); the topology is generated by open intervals of the following forms:

$$(-1,a) \qquad (a,1) \qquad [(-1,b)\setminus\{0\}]\cup\{0^*\} \qquad [(c,1)\setminus\{0\}]\cup\{0^*\},$$

   where $a\in(-1,1), b\in(0,1), c\in(-1,0)$. That is, the topology is generated by open intervals, where you choose exactly one of the two zeroes to put in.

   (a) Define two functions $f,g:(-1,1)\to X$, where $f(x)=x$ for all $x\in(-1,1)$ and $g(x)=x$ if $x\in(-1,1)\setminus\{x\}$ but $g(0)=0^*$. Show that both f and g are embeddings.

   (b) Show that X is T_1 but not Hausdorff.

   (c) Show that $f\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)$ and $g\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)$ are compact but not closed in X, and that their intersection is not compact.

3. Prove the locally compact version of the Tietze extension theorem.