Math 423 Homework 10

1. Show that if
$$F(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0, \end{cases}$$
then $F \notin BV([a,b])$ for any $a \leq 0 < b$ or $a < 0 \leq b$.

2. Show that $F : \mathbb{R} \to \mathbb{C}$ is Lipschitz with Lipschitz constant M if and only if F is absolutely continuous and $|F'| \leq M$ almost everywhere.

3. Let $F : (a,b) \to \mathbb{R}$ with $-\infty \leq a < b \leq \infty$; F is called **convex** if for all $s, t \in (a,b)$ and $\lambda \in (0,1)$,
$$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t).$$
   (a) Show that the definition of convexity given above is equivalent to: for all $s, t, s', t' \in (a,b)$ such that $s \leq s' < t'$ and $s < t \leq t'$,
   $$\frac{F(t) - F(s)}{t - s} \leq \frac{F(t') - F(s')}{t' - s'}.$$
   (b) Show that F is convex if and only if F is absolutely continuous on every compact subinterval of (a,b) and F' is increasing (on the set on which it is defined).
   (c) Show that if F is convex and $t_0 \in (a,b)$, then there is a $\beta \in \mathbb{R}$ such that $F(t) - F(t_0) \geq \beta(t - t_0)$ for all $t \in (a,b)$.
   (d) Prove Jensen's inequality: If $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$, $g : X \to (a,b)$ is in $L^1(\mu)$, and F is convex on (a,b), then
   $$F\left(\int g d\mu\right) \leq \int F \circ g d\mu.$$
   Hint: let $t_0 = \int g d\mu$ and $t = g(x)$ in the previous part, and integrate.