May 7, 2007

Syllabus in Differential Equations

1. Special functions and series solutions:
- Regular and singular points; Frobenius method;
- Bessel (cylinder) functions (differential equations; series solutions; integral representations)
- Orthogonal polynomials: Legendre/Jacoby; Hermite; Laguerre

N. N. Lebedev, Special functions and their applications, (ch. 4, 5)

2. Partial differential equations and systems
- 1-st order PDE: the method of characteristics (linear, quasi-linear, nonlinear).
- 2-nd and higher order equations: classification; initial/boundary-value problem; well-posedness; stability; dissipation; dispersion.

3. Separation of variables and generalized Fourier-series expansions:
- Sturm-Liuoville and multi-dimensional elliptic eigenvalue problems: orthogonality and completeness of eigenfunctions
- Laplace, heat and wave equations in bounded regions.

4. Transform methods (Fourier; Laplace; Henkel); problems in unbounded regions.

5. Green's functions and fundamental solutions:
- Green' identities;
- Generalized functions
- Fourier method: series and integral expansions of Green's functions
- Symmetries and the Method of images: Green's functions for the Laplace, heat and wave equations in special regions (space; half-space; quadrant; slab; box; disk; ball; sphere)

6. Variational, Perturbation and Asymptotic methods:
- Hamilton's principle of minimal action and applications: vibrating strings, membranes, etc. Energy conservation and causality.
- The mini-max principle and Rayleigh-Ritz method for eigenfunctions of differential operators;
- Regular perturbations (eigenvalue, boundary value perturbations)
- Equations with large parameter: Stationary phase and Geometrical optics methods; Helmholtz equation.

References:

R. Courant, D. Hilbert, Methods of Mathematical Physics
G.F.D. Duff and D. Naylor, Differential Equations of Applied Mathematics, John Wiley & Sons
A. Nayfeh, Perturbation Methods
I. Stakgold, Green's functions and boundary-value problems, Wiley-Interscience
E. Zauderer, Partial Differential Equations of Applied Mathematics, Wiley-Interscience
(reference to most basic topics)
W. A. Strauss, Partial Differential Equations: An Introduction, Wiley
G. Whitham, Linear and nonlinear waves (ch. 1,2,3,5,7,11)

Note: The above syllabus is centered on Partial Differential Equations. The material is partly covered in MATH 445 and MATH 448. Some topics require additional reading. Should this exam be offered in the future, the selection of topics may be modified, for example depending on the coursework of a particular student, or to ensure the breadth and non-overlap requirements. One such specific option would cover a subset of the topics above (roughly corresponding to MATH 445) plus a selection of Ordinary Differential Equations topics, most of which are included in the Dynamical Systems syllabus. (That option cannot be chosen should the student attempt also Dynamical Systems.)

The following pages contain a variant of the preceding syllabus including applications of Differential Equations to Mathematical Biology.

Syllabus in Differential Equations, Dynamical systems, Math. Biology

D. Gurarie

Partial Differential Equations

(based on Math 445 – 448, and additional topics)

1. Special functions and series solutions:
   - Regular and singular points; Frobenius method;
   - Bessel (cylinder) functions (differential equations; series solutions; integral representations)
   - Orthogonal polynomials: Legendre/Jacoby; Hermite; Laguerre

2. Partial differential equations and systems
   - 1-st order PDE: the method of characteristics (linear, quasi-linear, nonlinear).
   - 2-nd and higher order equations: classification; initial/boundary-value problem; well-posedness; stability; dissipation; dispersion.

3. Separation of variables and generalized Fourier-series expansions:
   - Sturm-Liouville and multi-D elliptic eigenvalue problems: orthogonality and completeness of eigenfunctions
   - Applications to Laplace, heat and wave equations in bounded regions.

4. Transform methods (Fourier; Laplace; Henkel); problems in unbounded regions.

5. Green's functions and fundamental solutions:
   - Green' identities;
   - Generalized functions
   - Series and integral expansions of Green's functions
   - Symmetries and reflections (Method of images) for Green's functions of Laplace, heat and wave equations in special regions (space; half-space; quadrant; slab; box; disk; ball; sphere)

6. Variational and Perturbation methods.
   - The Hamilton’s principle of minimal action and application: vibrating strings, membranes, etc.
   - Energy conservation and causality.
   - The mini-max principle and Rayleigh-Ritz method for eigenfunctions of differential operators;
   - Regular perturbations (eigenvalue, boundary value perturbations)

References:

W. A. Strauss, Partial Differential Equations: An Introduction, Wiley
G.F.D. Duff and D. Naylor, Differential Equations of Applied Mathematics, John Wiley & Sons
I. Stakgold, Green's functions and boundary-value problems, Wiley-Interscience
E. Zauderer, Partial Differential Equations of Applied Mathematics, Wiley-Interscience
(reference to most basic topics)
G. Whitham, Linear and nonlinear waves (ch. 1,2,3,5,7,11)
Lebedev, Special functions and their applications, (ch. 4, 5)

Dynamical Systems

1. Ordinary differential equations and systems

- Linear systems (eigenvalue method).
- Equilibria, stability, Bifurcation (Routh-Hurwitz; periodic orbits, stable/ unstable orbits)
- Poincare-Bendixon theorem

2. *Discrete dynamical systems*

- Periodic points: stability, hyperbolicity; linearization;
- Bifurcations

J. Guckenheimer & P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations*
R.C. Robinson, *Dynamical Systems: stability, symbolic dynamics, and chaos*
V. I. Arnold, Ordinary Differential Equations

Mathematical Biology (Math 449)

1. **Population dynamics**: discrete and continuous models, aging structure, metapopulations, interacting species (competition, predation, parasitism): equilibria, stability, phase-plane analysis; bifurcations
2. **Infectious disease modeling**: SIR/SEIR (persistence, eradication, control), metapopulation models; Vector-borne diseases; Macro-parasite diseases; Individual-based models
3. **Population genetics and evolution**: selection and mutation; Evolution of genetic systems
4. **Biological motion**: Models of chemotaxis, Biological invasion, Traveling wave solutions of reaction-diffusion equations
5. **Pattern formation**: Turing instability and bifurcations in activator-inhibitor systems

N.F. Britton, Essential Mathematical Biology, Springer, 2003
J. D. Murray, Mathematical Biology, Springer, 2002
L. Edelstein-Keshet, Mathematical models in Biology, SIAM, 2005
R.M. Anderson, R.M. May, Infectious diseases of humans, Oxford UP, 1998;
M. Novak and R. May "Viral dynamics: the mathematical foundations of virology and immunology", by, Oxford UP, 2000
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